M. A./M. Sc. Examination-2020

Semester-III

Mathematics

MMC 33 (Fluid Mechanics)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) What is the significance of the equation of continuity in fluid mechanics? [1]

   (b) Show that Eulerian and Lagrangian forms of equations of continuity are equivalent. [3]

   (c) The velocity components of a flow in cylindrical polar coordinates are $(trz\sin\theta, trz\cos\theta, -\frac{1}{2}tz^2\sin\theta)$. Determine the components of the acceleration of a fluid particle. [3]

   (d) Assuming that the velocity components for a two dimensional flow system can be given in the Eulerian system by $u = A(x+y)+ct,\ v = Bx - Ay + Et$, find the displacement of a fluid particle in the Lagrangian system. [3]

2. (a) Derive the differential equation that must be satisfied by a boundary surface of a fluid in motion. [3]

   (b) Check whether the ellipsoid

$$\frac{x^2}{a^2e^t\sin(t+\frac{\pi}{4})} + \frac{y^2}{b^2e^t\cos(t+\frac{\pi}{4})} + \frac{z^2}{c^2\sec 2t} = 1$$

   is a possible form of boundary surface of a liquid for any time t. Find the normal velocity of the boundary, if it is a boundary form. [4]

   (c) Define stream line, path line and vortex line. Find their differential equations.

3. (a) Show that if the motion of an ideal fluid, for which density is a function of pressure p only is steady and the external forces are conservative, then there exists a family of surfaces which contain the stream lines and vortex lines. [3]

   (b) For a steady motion of inviscid incompressible fluid of uniform density under conservative forces, show that the velocity $\vec{r}$ and velocity $\vec{q}$ satisfy

$$(\vec{q}\cdot\vec{\nabla})\vec{r} = (\vec{r}\cdot\vec{\nabla})\vec{q}.$$

   [2]

   (c) Find the curvature at any point of a streamline. [2]

   (d) Show that the velocity potential

$$\phi = \frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$

   gives a possible motion. Determine the stream line, and show also that the curves of equal speed are the ovals of Cassini given by $rr' =$ constant where r, r' are the distances of the point $P(x,y)$ from the points $(a,0)$ and $(-a,0)$. [3]

4. (a) State and prove the Milne-Thomson circle theorem in fluid mechanics. [2+2]

   (b) Find the image of a doublet with regard to a circle. [3]

   (c) A circular cylinder is placed in a uniform stream. find the forces acting on the cylinder. [3]

5. (a) State and prove the Kelvins minimum energy theorem. [2+3]

   (b) Show that if $\phi = -\frac{ax^2+by^2+cz^2}{2},\ v = \frac{lx^2+my^2+nz^2}{2}$ where a,b,c,l,m,n are functions of time and $a+b+c=0$, irrotational motion is possible with a free surface of equipressure if $(l+a^2+\dot{a})e^{2\int adt}$, $(m+b^2+\dot{b})e^{2\int bdt}$ and $(n+c^2+\dot{c})e^{2\int cdt}$ are constants. [5]

6. (a) Find the complex potential for the motion of a circular cylinder moving in an infinite mass of the liquid at rest at infinity with velocity U in the direction of x-axis. Find the stream lines and equipotential lines. [3+1+1]

   (b) Find the velocity distribution in the generalized Coutte flow. Determine shearing stress, skin friction and the coefficient of friction at both the walls. [2+1+1+1]

Use separate answer script for each unit

M. A./M. Sc. Examination-2020

Semester-III
Mathematics
Paper: MMC-34(New Syllabus)
(Calculus of Variations and Special Functions)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Marks : 20)
(Calculus of Variations)

Answer *any two* questions.

1. (a) Show that the Euler-Lagrange equation of the functional $\int_{x_1}^{x_2} F(x, y, y', y'')dx$ with $y(x_1) = y_1$, $y(x_2) = y_2$ has the first integral $F_{y'} - \frac{d}{dx}F_{y''}=$ constant, if $F_y = 0$, where the 'dash' denotes differentiation with respect to x. [4]

   (b) Find the stationary function and stationary value of the following functional
   $I[y(x)] = \int_0^1 (y'^2 + xy)dx, \quad y(0) = 0, y(1) = 1.$ [4]

   (c) Discuss briefly regarding 'Natural Boundary Condition' in a variational problem. [2]

2. (a) Find the curve joining given points A and B which is traversed by a particle moving without friction under gravity from A to B in the shortest time. [4]

   (b) Find the curve with fixed point boundary points such that its rotation about the axis of abscissae gives rise to a surface of revolution of minimum surface area. [4]

   (c) If $y(x)$ is an extremizing function for $I(y) = \int_{x_1}^{x_2} F(x, y, y')dx$ with $y(x_1) = y_1$, $y(x_2) = y_2$, then show that the first variation $\delta I(y) = 0$. [2]

3. (a) Find the extremal of the functional
   $I(y) = \int_0^{x_1} y'^2 dx$ with $y(0) = 0$, $y(x_1) = \frac{2}{1-x_1}$. [5]

   (b) Find the extremum the functional $I[y(x)] = \int_0^1 (y'^2 + y^2)dx$ with $y(0) = 0$, $y(1) = 1$ using the Rayleigh-Ritz method. Also, prepare a table to compare the results obtained from exact solution and two-term approximate solution upto three decimal places accuracy. [5]

Unit-II [Special Functions(Marks: 20)]

Answer *any two* questions.

1. (a) Using Frobenius method, obtain the indicial equation and recurrence relation of Bessel's differential equation. Also obtain the Weber's Bessel function of order zero. [3+2]

   (b) Show that $G(x,t) = \frac{1}{1-t}e^{\frac{-xt}{1-t}}$ is a generating function for Laguerre polynomial.
   Hence show that $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$. [3+2]

2. (a) Show that Hermite polynomials are orthogonal over $(-\infty, \infty)$ with respect to the weight function e^{-x^2}. [5]

   (b) Prove the following recurrence relations for Legendre Polynomials
   $nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$,
   $nP_n = xP_n' - P_{n-1}'$,
   $(2n+1)P_n = P_{n+1}' - P_{n-1}'$. [5]

3. (a) Define regular singular point and irregular singular point of a 2nd order ordinary differential equation.
   How a singular point can be extended to include the point at infinity? [2+2]

(b) What is a hypergeometric function? Find its derivative. Write down the integral representation of hypergeometric function. [3]

(c) Define Dirac delta function $\delta(x - x_0)$ in integral form. Explain the scaling property of this function. [1+2]

M. A./M. Sc. Examination-2020
Semester-III
Mathematics
Paper: MMO-31 (A6)(New Syllabus)
(Dynamical Meteorology)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Define the potential temperature and obtain the relation $S = C_p l_n \theta$ +constant, between the entropy S and potential temperature θ of a parcel of dry air. Show that the potential temperature remains constant in an isentropic flow. [1+3+1]

   (b) Derive the following relations for a reversible saturated adiabatic changes of state in the atmosphere: $\left(\frac{\partial T}{\partial p}\right)_{moist} < \left(\frac{\partial T}{\partial p}\right)_{dry}$. [5]

2. (a) State the assumptions of geostrophic balance approximation of the atmosphere. Obtain the geostrophic relationship in isobaric coordinates. Deduce the thermal wind equation in the form

$$p\frac{\partial \vec{V_g}}{\partial p} = -\frac{R}{f}\,\hat{k} \times (\nabla T)_p.$$ [6]

   (b) Show that the Cyclostrophic balance approximation is valid for large Rossby number. [2]

   (c) For an ideal gas, derive the relation $p = R\, e^{(S-S_0)/C_v} \rho^\gamma$, where ρ is the density of dry air and γ is the ratio of C_p and C_v. [2]

3. (a) Obtain the hypsometric equation. Deduce the expression for dry adiabatic lapse rate of temperature with altitude. Discuss the stability of the atmospheric equilibrium. [5]

   (b) Show that the rate of change of absolute circulation C round a closed curve Γ moving with the atmosphere is given by $\dot{C} = \int\int_\sigma grad\, T \times grad\, S.\hat{n}\, d\sigma$, where σ is an open suface, T and S are the absolute temperature and specific entropy, and $\hat{n}$ is the unit outward normal to σ. [5]

4. (a) Obtain the general equation of motion of the atmosphere relative to the rotating earth and hence deduce the Euler's equation of motion for a perfect fluid, and Navier-Stokes equation for the viscous fluid. [6]

   (b) Define the potential vorticity of the atmosphere. Obtain Ertel's formula for potential vorticity in an inviscid rotating fluid and establish Rossby's theorem. [4]

5. (a) Discuss the development of sea-breeze circulation. [5]

   (b) Starting from the thermodynamic energy equation, obtain the Richardson's equation by using quasi-static equation of continuity. [5]

6. (a) Discuss the events that leads to the development of a tropical cyclone. [5]

   (b) Write short notes on the development of Lee waves and determine the mathematical equation associated with such type of waves. [5]

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Semester-III

Mathematics

Optional Course : MMO-31 (A9) (New)

(Mathematical Pharmacology-I)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. Define the efficacy of a drug. Using rate theory, prove that the equilibrium response $\gamma = \phi \frac{k_2 x}{x + k_2/k_1}$, k_1 is the association rate constant, k_2 is the dissociation rate constant, x is the concentration of drug applied and ϕ is a constant. [3+7]

2. Derive the equation for simple monovalent cell surface binding in case of constant ligand concentration. Nondimensionalise the equation and hence solve it. [3+7]

3. Define specific and non-specific binding. Derive the equation for the specific-binding and solve it. [3+7]

4. Derive the model equations which exhibit interconversions of receptor and complex states. Assuming equilibrium binding of ligand to both receptor forms, obtain the transient solutions for both complexes. [3+7]

5. Write down the base model for endocytosis. Obtain the ratio of total number of surface receptor in the presence of ligand to that in the absence of ligand. [3+7]

6. Derive equation of mass conservation. Suppose a bolus of N molecules of A is injected into a long cylinder at $t = 0$ so that all the molecules are present within an infinitesimal volume at $x = 0$ and applying a perfectly sink condition at $x \to \pm\infty$, obtain the expression for the concentraton of A (C_A) at time t. [5+5]

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Semesester-III

Mathematics

Optional Course: MMO-31(P-05)

(Algebraic Coding Theory-I)

Time: Three Hours — Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer *any four* questions.

1. (a) Define communication channel. Let $C = \{001, 011\}$ be a binary code. Suppose we have a memoryless binary channel with the following probabilities: $P(0 \text{ received}|0 \text{ sent}) = 0.1$ and $P(1 \text{ received}|1 \text{ sent}) = 0.5$. Use the maximum likelihood decoding rule to decode the received word 000. [2+4]

   (b) For a BSC with crossover probability $p < 1/2$, show that the maximum likelihood decoding rule is the same as the nearest neighbour decoding rule. [4]

2. (a) Design a channel coding scheme to correct one error for the message source

$$\{000, 100, 010, 001, 110, 101, 011, 111\}.$$

   Can you find one of the best schemes in terms of information transmission speed? [4+2]

   (b) For some $m \geq 1$, show that each cyclotomic coset q modulo $q^m - 1$ contains at most m elements. [4]

3. (a) Let C be a linear code of length n over F_q. Then show that $C^{\perp}$ is a linear code and $\dim(C) + \dim(C^{\perp}) = n$. [4+1]

   (b) Show that an $(n-k) \times n$ matrix H is a parity-check matrix for C if and only if the rows of H are linearly independent and $HG^T = O$. [5]

4. (a) Let $C = \{0000, 1011, 0101, 1110\}$ be a linear code over F_2. Then decode the received words 1101 and 1111. [2+2]

   (b) Show that every codeword in a self-orthogonal binary code has even weight. [4]

   (c) Let C be the binary $[5, 3]$-linear code with the generator matrix

$$G = \begin{pmatrix} 10110 \\ 01011 \\ 00101 \end{pmatrix}.$$

   Find the information rate of C. [2]

5. (a) Show that unitary modules over $\mathbb{Z}$ are only abelian groups. [4]

   (b) Show that direct sum of free modules is a free module. [4]

   (c) Give an example of a non-free module. [2]

6. (a) Give an example with proper justification of a free module in which a linearly independent subset can not be extended to a basis. [4]

   (b) Show that an unitary module over a ring with unity is a quotient of a free module. [4]

   (c) State Schur's Lemma of modules. [2]

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Semester-III

Mathematics

Paper: MMO 31 (P01) (New)

(Advanced Complex Analysis-I)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Prove that any analytic function defined in a domain D has the mean value property in D. Is it true for any harmonic function defined in D ? Justify your answer. [3]
   (b) State and prove Poisson's integral formula. [4]
   (c) Let $f(z)$ be regular in the closed disc $|z| \leq R$ and let $v(r,\theta)$ be its imaginary part. If $v(r,\theta) \geq 0$ throughout the disc, then prove that $\frac{1}{2}v(0,0) \leq v(r,\theta) \leq 2v(0,0)$ $[0 \leq \theta \leq 2\pi, 0 \leq r < R]$. [3]

2. (a) State Dirichlet's problem for a disc in case of harmonic functions. Discuss the existence and uniqueness of the solution of the problem. [6]
   (b) Let $u(x,y)$ be harmonic in a domain D and continuous in $\overline{D}$. If $u(x,y) = \lambda$, a constant on the boundary of D, then show that $u(x,y) = \lambda$ in D. [2]
   (c) Show that $u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\phi}-z|^2} u(R,\phi)d\phi$. [2]

3. (a) Prove that for a non-constant entire function $f(z)$, its maximum modulus function $M(r,f) \to \infty$ as $r \to \infty$. [2]
   (b) Show that for any transcendental entire function $f(z)$, $limsup_{r\to\infty} \frac{logM(r,f)}{logr} = \infty$. [2]
   (c) State and prove Hadamard's three circles theorem. [6]

4. (a) Let $f(z)$ be a non-constant analytic function in $|z| \leq R$ with $f(0) = 0$. Then show that for $0 \leq r < R, M(r) < \frac{2R}{R-r}A(R)$, where $M(r), A(r)$ denote the maximum value of $|f(z)|, Rlf(z)$ respectively on $|z| = r$. [4]
   (b) If $f(z)$ is analytic for all finite values of z and $A(r) = O(r^k), k > 0$, then prove that $f(z)$ is a polynomial of degree not exceeding k. [3]
   (c) If $f(z)$ is a transcendental entire function of order ρ and $P(z)$ is a non-constant polynomial then show that the product $f(z).P(z)$ is of order ρ. [3]

5. (a) Prove that a transcendental entire function and its derivative have the same order. [4]
   (b) If $f(z)$ is an integral function of finite order ρ and $r_1, r_2, \ldots\ldots\ldots, r_n, \ldots\ldots$ are the moduli of the zeros of $f(z)$ then show that $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ converges if $\alpha > \rho$. [3]
   (c) Show that the exponent of convergence of zeros of $cos\ 2z$ is 1 [3]

6. (a) If $f(z)$ is an entire function of order ρ then show that for every $\epsilon > 0$, the inequality $n(r) < r^{\rho+\epsilon}$ holds for all sufficiently large r, where $n(r)$ denotes the number of zeros of $f(z)$ in the closed disc $|z| \leq r$. [3]
   (b) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of finite non-zero order ρ, then find the expression for ρ in terms of a_n. [5]
   (c) Find the type of $\sum_{n=1}^{\infty} \frac{z^n}{(n!)^{\beta}}$, $\beta > 0$. [2]

M. A./M. Sc. Examination-2020

Semester-III

Mathematics

Optional Course : MMO-31 (P03)(New Syllabus)

(Advanced Real Analysis-I)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer **question no.6** and any **three** from the rest

1. (a) Show that the set of all points of discontinuities of any function is a F_σ-set. [3]
   (b) Define lim sup and lim inf of a function f at a point a. If one of the Dini derivates of a continuous function f is continuous at x then show that $f'(x)$ exists. [2+4]
   (c) Show that for any set E the set of all isolated points is countable [3]
2. (a) State and prove Zygmond theorem [4]
   (b) Define Cantor function on $[0, 1]$ and show that it is continuous and non decreasing on $[0, 1]$. [2+3+3]
3. (a) Show that an absolutely continuous function is continuous, is of bounded variation and satisfies Lusin condition. [2+3+4]
   (b) If f is absolutely continuous on R and $D^+f \geq 0$ a.e then f is non-decreasing. [3]
4. (a) Define point of outer density of a set and approximate continuity of a function. State and prove Lebesgue density theorem. [2+5]
   (b) Show that the class of approximate continuous functions forms a linear space. [5]
5. (a) Show that an usc function in [a,b] is bounded above and attains its upper bound. [4]
   (b) Let a sequence of Baire class 1 functions $\{f_n\}$ converges to f. Show that f is also a Baire class 1 function. [6]
   (c) What is tagged partition, Define Haustock integral of a function. [2]
6. Answer any two
   (a) Give example of a Baire 2 function which is not Baire 1. [2]
   (b) If x is a point of disparsion of E then show that x is a point of outer density of E^c. [2]
   (c) Show that a continuous function is approximate continuous.. [2]

M. A./M. Sc. Examination-2020

Semester-III

Mathematics (Advanced Functional Analysis)

Course: MMO-31(P2)

Time: 3 Hours

Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) If K and C are compact and closed sets respectively in a topological vector space(TVS) X such that $K \cap C = \phi$, show that there is a neighbourhood V of θ in X such that $(K+V) \cap (C+V) = \phi$. Hence deduce that every topological vector space is regular. [6]

   (b) Define absorbing set in a TVS X. Prove that every neighbourhood of θ in X contains an absorbing neighbourhood of θ in X. [3]

   (c) If X is locally bounded TVS with Heine-Borel property, then prove that X is finite dimensional. [1]

2. (a) Let X be a TVS. If X is metrizable by an invariant metric d then prove that
   (i) $d(nx, \theta) \leq nd(x, \theta)$, $\forall x \in X$ and $n \in \mathbb{N}$
   (ii) if $\{x_n\}$ is a sequence in X such that $x_n \to \theta$ as $n \to \infty$ then there exist a sequence $\{\gamma_n\}$ of scalars such that $\gamma_n \to \infty$ and $\gamma_n x_n \to \theta$ as $n \to \infty$. [4]

   (b) When a mapping is said to be uniformly continuous on a TVS? If X and Y are TVS over the same scalar field and $T : X \to Y$ is a linear operator then show that T is continuous if and only if T is continuous at θ in X. [4]

   (c) Prove that every compact subset of a TVS is bounded. [2]

3. (a) Prove that every locally compact subspace of a TVS is closed. [4]

   (b) Let X be a linear space and A be a convex and absorbing set in X containing θ. Then show that
   (i) μ_A is a seminorm if A is balanced set in X.
   (ii) if $B = \{x \in X : \mu_A(x) < 1\}$ and $C = \{x \in X : \mu_A(x) \leq 1\}$ then $B \subset A \subset C$ and $\mu_A = \mu_B = \mu_C$. [4]

   (c) For any seminorm p on a vector space X over $\mathbb{K}$, prove that the set $\{x \in X : p(x) \leq 1\}$ is convex, absorbing and balanced. [2]

4. (a) Let $\{T_n\}$ be a sequence of continuous linear functions defined from a TVS X to a TVS Y.
   (i) If the set $A = \{x \in X : \{T_n(x)\}\ is\ cauchy\}$ is of second category in X then prove that $X = A$.
   (ii) If the set $B = \{x \in X : \{T_n(x)\}\ is\ convergent\}$ is of second category and Y is an F-space then prove that $X = B$ and if $T(x) = \lim_{n\to\infty} T_n(x)$, $x \in X$ then T is continuous. [6]

   (b) If Y is a subspace of a TVS X and Y is an $F-$space (in the topology inherited from X), then prove that Y is a closed subspace of X. [4]

5. (a) State and prove Hanh-Banach Theorem for complex linear space(assuming the corresponding results of real linear space). [4]

   (b) Let A and B be disjoint, non-empty, convex sets in a TVS X. If A is open then show that there exist $T \in X^*$ and $\gamma \in \mathbb{R}$ such that $RlT(x) < \gamma \leq RlT(y)$ for every $x \in A$ and $y \in B$. [4]

   (c) Let B be a convex balanced local base in a TVS X and for every $V \in B$, let μ_V be its Minkowski functional. Prove that $\{\mu_V : V \in B\}$ is separating family of continuous seminorms on X. [2]

6. (a) State and prove Banach-Alagulu Theorem. [7]

   (b) Let (X, τ) be a TVS, N be a closed subspace of X and τ_N be the quotient topology on X/N induced by the mapping $\pi : X \to X/N$ where $\pi(x) = x + N$. Then show that $(X/N, \tau_N)$ is a TVS. [3]

M. A./M. Sc. Examination-2020
Semester-III
Mathematics
MM0-31(A7/P8)(New)
(Lie Theory of Ordinary and Partial Differential Equations)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. Define Lie group and Lie group of transformations. Prove that set of all 2×2 matrices $\begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix}, \epsilon \in [-\pi, \pi]$ forms a Lie group under some binary composition in a domain for ϵ to be specified by you. Verify whether the set of transformations $(x, y) \to (x^*, y^*) = \mathcal{T}_\epsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ is a Lie group of transformations. [2+3+5]

2. What do you mean by Lie group of transformations in its infinitesimal form. Define corresponding infinitesimal generator for the Lie group of transformations in a geometric space of dimension n. Find infinitesimal generators for the Lie group of transformations $(x, y) \to (x^*, y^*) = \left(\frac{x+\epsilon_1}{1+\epsilon_3 x+\epsilon_4 y}, \frac{y+\epsilon_2}{1+\epsilon_3 x+\epsilon_4 y}\right)$. Find the commutator table for the infinitesimal generators of the Lie group of transformations mentioned above. [1+2+3+4]

3. Define canonical variables and invariant function in the geometric space involved with a Lie group of point transformations. State principles for their determination. Find canonical variables for the Lie group of point transformations $(x, y) \to (x^*, y^*) = (x \cos\epsilon + y \sin\epsilon, -x \sin\epsilon + y \cos\epsilon)$. State whether any one of these canonical variables is an invariant function for the Lie group of point transformations mentioned above. [2+2+5+1]

4. Define Lie algebra. When a Lie algebra is called solvable? Show that set of infinitesimal generators for the Lie group of point transformations $x \to x^* = \frac{(1+\epsilon_1)x+\epsilon_2}{1-\epsilon_3 x}$ follow all conditions for a Lie algebra. Verify whether the Lie algebra just obtained is solvable? [1+1+6+2]

5. Verify which of the following two equations

$$a)\ y''(x) + p(x)y'(x) + q(x)y(x) = 0, \qquad b)\ y''(x) + p(x)y'(x) + q(x)y(x) = \psi(x),$$

$p(x), q(x), \psi(x) \in C(\mathbb{R})$ admits $x \to x^* = x,\ y \to y^* = e^\epsilon y$ as the Lie group of symmetry transformation by

i) the method of direct substitution,

i) the invariant function of second order prolonged geometric space.

[5+5]

6. Find infinitesimal generators for the Lie group of point transformations admitted by the heat Eq. $u_t = u_{xx}$. [10]

M. A./M. Sc. Examination-2020

Semester-III

Mathematics

MMO 31 : Optional Paper

Rings and Modules-I

Time: Three Hours | **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Every ring is a ring with 1.

Answer *any four* questions.

1. (a) Suppose that the diagram of R-modules and R-morphisms

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' \end{array}$$

is commutative and has exact rows. If α, γ and g are epimorphisms then show that β is so. [3]

(b) Let M be an R-module. Define annihilator $ann_R(M)$ of M in R. Show that M is a faithful $R/ann_R(M)$-module. [3]

(c) Let R be a ring. Prove that R is a simple R-module if and only if it is a division ring. [2]

(d) Give examples of subspaces A, B, C of $\mathbb{R}^2$ such that $(A \cap B) + (A \cap C) \neq A \cap (B + C)$. [2]

2. (a) State and prove the universal mapping property of the direct product of a family of R-modules $\{M_\alpha\}$. [4]

(b) Let A, B be two submodules of an R-module M. Establish an exact sequence of the form

$$0 \longrightarrow M/A \cap B \longrightarrow M/A \times M/B \longrightarrow M/A + B \longrightarrow 0.$$

Deduce that $(A + B)/(A \cap B) \simeq (A + B)/A \times (A + B)/B \simeq A/(A \cap B) \times B/(A \cap B)$. [3]

(c) Let $f : M \longrightarrow N$ be an R-morphism. For every submodule B of N, show that $f(f^{-1}(B)) = B \cap Im\ f$. [3]

3. (a) Prove that a short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ splits on the left if and only if $Im\ f = \ker f$ is a direct summand of B. [4]

(b) Show that every external direct sum $M = \oplus M_\alpha$ is isomorphic to an internal direct sum of a family $\{N_\alpha\}$ of suitable submodules of M. [3]

(c) Let $\{M_\alpha\}$ be a family of R-modules. Consider the family of canonical injections $\{i_\beta : \oplus M_\alpha \longrightarrow M_\beta\}$. Then prove that the canonical projection $\pi_\beta : \oplus M_\alpha \longrightarrow M_\beta$ is the unique R-morphism such that

$$\pi_\beta \circ i_\alpha = \begin{cases} Id_{M_\beta} & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

4. (a) Define subdirect product of rings. Show that the ring of integers is a subdirect product of finite fields. [1+2]

(b) Show that every ring is a subdirect product of subdirectly irreducible rings. [3]

(c) State and prove a necessary and sufficient condition for an R-module M to be a subdirect product of a family of R-modules $\{M_\alpha\}$. [4]

5. (a) If an R-module F satisfies the universal mapping property, then show that F is free. [3]

(b) A left R-module F is free over X if and only if $R \simeq \oplus_X Rx$. [3]

(c) Show that every commutative ring with 1 is an IBN ring. [4]

6. (a) Show that for every R-modules M_R and ${}_RN$, a tensor product of M and N exists. [4]

(b) Let M_α be a family of right R-modules and N be a left R-module. Show that $(\oplus M_\alpha) \otimes N \simeq \oplus(M_\alpha \otimes N)$. [3]

(c) Let I be an ideal of R and M be a right R-module. Show that $M \otimes R/I \simeq M/MI$. [3]

M. A./M. Sc. Examination-2020

Semesester-III

Mathematics

Optional Course: MMO-31(P-05)

(Algebraic Coding Theory-I)

Time: Three Hours Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any four*** questions.

1. (a) Define communication channel. Let $C = \{001, 011\}$ be a binary code. Suppose we have a memoryless binary channel with the following probabilities: $P(0 \text{ received}|0 \text{ sent}) = 0.1$ and $P(1 \text{ received}|1 \text{ sent}) = 0.5$. Use the maximum likelihood decoding rule to decode the received word 000. [2+4]

   (b) For a BSC with crossover probability $p < 1/2$, show that the maximum likelihood decoding rule is the same as the nearest neighbour decoding rule. [4]

2. (a) Design a channel coding scheme to correct one error for the message source

$$\{000, 100, 010, 001, 110, 101, 011, 111\}.$$

   Can you find one of the best schemes in terms of information transmission speed? [4+2]

   (b) For some $m \geq 1$, show that each cyclotomic coset q modulo $q^m - 1$ contains at most m elements. [4]

3. (a) Let C be a linear code of length n over F_q. Then show that $C^\perp$ is a linear code and $\dim(C) + \dim(C^\perp) = n$. [4+1]

   (b) Show that an $(n-k) \times n$ matrix H is a parity-check matrix for C if and only if the rows of H are linearly independent and $HG^T = O$. [5]

4. (a) Let $C = \{0000, 1011, 0101, 1110\}$ be a linear code over F_2. Then decode the received words 1101 and 1111. [2+2]

   (b) Show that every codeword in a self-orthogonal binary code has even weight. [4]

   (c) Let C be the binary $[5, 3]$-linear code with the generator matrix

$$G = \begin{pmatrix} 10110 \\ 01011 \\ 00101 \end{pmatrix}.$$

   Find the information rate of C. [2]

5. (a) Show that unitary modules over $\mathbb{Z}$ are only abelian groups. [4]

   (b) Show that direct sum of free modules is a free module. [4]

   (c) Give an example of a non-free module. [2]

6. (a) Give an example with proper justification of a free module in which a linearly independent subset can not be extended to a basis. [4]

   (b) Show that an unitary module over a ring with unity is a quotient of a free module. [4]

   (c) State Schur's Lemma of modules. [2]

M. A. / M. Sc. Examination - 2020

Semester-III

Mathematics

Paper: MMO 31 (A05) (New)

(Dynamics of Ecological System-I)

Time: 3 Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. Analyse a logistic model for a single species non-age structured population. Show that the population growth curve possesses a point of inflexion when half the final population size is attained. Find also the time at which the point of inflexion occurs. [4+3+3]

2. Consider the following competing species model:

$$\begin{aligned} \frac{dx}{dt} &= x(3-2y-x), \\ \frac{dy}{dt} &= y(2-x-y), \\ x(t=0) &\geq 0, \ y(t=0) \geq 0. \end{aligned}$$

(a) Determine and classify all equilibria.
(b) Sketch the phase portrait using the linear systems.
(c) Sketch the phase portrait using the method of nullclines.
(d) What happens to the competing populations as time increases? [3+3+2+2]

3. (a) Consider the following modified Lotka-Volterra predator-prey model:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right) - \alpha xy, \\ \frac{dy}{dt} &= \beta xy - dy, \\ x(t=0) &\geq 0, \ y(t=0) \geq 0, \end{aligned}$$

where all the system parameters are positive. Construct a Lyapunov function for which the co-existence equilibrium point is globally asymptotically stable (GAS).

(b) Consider a Holling type II Bazykin's interacting species model

$$\begin{aligned} \frac{dN}{dt} &= aN\left(1-\frac{N}{K}\right) - \frac{bNP}{\alpha+N}, \\ \frac{dP}{dt} &= -cP + \frac{cbNP}{\alpha+N} - dP^2, \ 0<c<1, \\ N(t=0) &\geq 0, \ P(t=0) \geq 0, \end{aligned}$$

where all the system parameters are positive. Put your comments on the system saddle-node bifurcation around the coexistence equilibrium point. [5+5]

4. (a) What do you mean by prey refuge in an interacting species system?
(b) Comment on the influence of the refuge parameter m and the fear factor k about the co-existence equilibrium point of the following Beddington-DeAngelis predator-prey model:

$$\begin{aligned} \frac{du}{dt} &= \frac{ru}{1+ku} - du - eu^2 - \frac{puv}{(1-m)u+\alpha v+\beta}, \\ \frac{dv}{dt} &= \frac{epuv}{(1-m)u+\alpha v+\beta} - \gamma v, \ 0<e<1, \ 0\leq m<1, \\ u(t=0) &\geq 0, \ v(t=0) \geq 0, \end{aligned}$$

where all the system parameters are positive. [2+(4+4)]

5. (a) What do you mean by SIR model for spread of disease?
(b) Explore the Kermack-McKendrick threshold theorem for SIR epidemic model. [2+8]

6. (a) Consider an infectious disease for which there is no recovery, such as untreated tuberculosis (Mycobacterium Tuberculosis) or many plant diseases. A simple transmission model for such a disease which includes vital rates and disease caused mortality is given by

$$\begin{aligned} \frac{dS}{dt} &= N\mu - \mu S - \beta SI, \\ \frac{dI}{dt} &= \beta SI - \mu I - \delta I, \\ S(t=0) &\geq 0, \; I(t=0) \geq 0. \end{aligned}$$

Locate the equilibria and create their stability in respect of basic reproductive number R_0.
(b) An eco-epidemiological model with disease in the prey species is given by

$$\begin{aligned} \frac{dx_s}{dt} &= r_1 x_s - b x_s(x_s + x_i) - \beta x_s x_i, \\ \frac{dx_i}{dt} &= \beta x_s x_i - \frac{\alpha x_i y}{a + x_i} - \delta x_i, \\ \frac{dy}{dt} &= (r_2 - \frac{\gamma y}{a + x_i})y, \\ x_s(t=0) &\geq 0, \; x_i(t=0) \geq 0, \; y(t=0) \geq 0, \end{aligned}$$

where all the system parameters are positive. Interpret the effect of disease on prey in the absence of predators and disease effects on predators in the absence of susceptible prey species. [5+5]

M.A./M.Sc. Examination - 2020
Semester - III
Mathematics
Course: MMO - 31(P-11/A-13)
(Theory of Computation - I)

Time: Three Hours Full Marks: 40

Questions are of value as indicated in the margin
Notations and symbols have their usual meanings

Answer *any four* questions

1. (a) Define homomorphism between sequential machines. Suppose,

$$\phi : S \to T$$

is a homomorphism between two machines S and T. Define a right congruence relation R on S such that T is isomorphic to S/R. [5]

(b) Define equiresponse relation ρ on S. Define the quotient sequential machine $modulo-\rho$. Show that the quotient machine is isomorphic to S^c, the connected submachine of S. [5]

2. (a) Let R_1 and R_2 be two right congruence relations on Σ_k^* which respectively refines β_1, $\beta_2 \subseteq \Sigma_k^*$. Show that there exists a homomorphism from $T(R_1, \beta_1)$ into $T(R_2, \beta_2)$ if and only if $R_1 \subseteq R_2$ and $\beta_1 \subseteq \beta_2$. [5]

(b) Define the minimal machine S^M associated with a given sequential machine S with output. Show that S^M is well-defined and has the same behaviour as S. Show also that there exists a strong homomorphism from S onto S^M. [1+2+2]

3. (a) Let S and T be two finite machines having m and n states respectively. Show that $S \equiv T$ if and only if $a \equiv^{m+n-2} b$ where a and b are respectively the initial states of S and T. [3]

(b) Define actual probabilistic machine and an isolated cut-point in connection with probabilistic machine. Show that if S is an actual probabilistic machine having isolated cut-point λ then $\beta_S(\lambda)$ is a definite set. Comment on the converse proposition. [1+1+4+1]

4. (a) Define Prenex normal form and Skolem standard form in mechanical theorem proving. Transform

$$F_1 : (\exists x)\{P(x) \wedge (\forall y)(D(y) \to L(x,y))\}$$
$$F_2 : (\forall x)\{P(x) \to (\forall y)(Q(y) \to \sim L(x,y))\} \text{ and}$$
$$G : (\forall x)(D(x) \to \sim Q(x))$$

into a set S of clauses. [1+1+2]

(b) Consider a set $S = \{P(x), \sim P(f(y))\}$ of clauses. Is it possible to find an interpretation that satisfies S? [1]

(c) Show that a set S of clauses is unsatisfiable if and only if there is a finite set of ground instances of clauses of S that is unsatisfiable. [5]

5. (a) Show that if S represents the Skolem standard form of a formula F then F is inconsistent if and only if S is so. [4]

(b) State and prove Davis and Putnam's pure-literal rule. [1+2]

(c) Define resolvent of clauses. Show that a resolvent C of clauses C_1 and C_2 is a logical consequence. [1+1]

(d) Check whether the set $\{P(x, f(y,z)), P(u, f(a,b)), P(x, f(g(v),b))\}$ is unifiable. [1]

6. (a) Show that a set S of clauses is unsatisfiable if and only if there is a resolution deduction of empty clause from S. [5]

(b) Use Robinson's resolution principle to establish the validity of the argument:

Custom officials searched everyone who entered this country and who was not a VIP. Some of the drug-pushers entered this country and they were only searched by the drug-pushers. No drug-pusher was a VIP. Therefore, some of the officials were drug-pushers.

[5]