B. Sc. Honours Examination-2020

Semester-I(CBCS)
Mathematics
Core course:CC-1
(Analysis-I)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any six*** questions.

1. (a) State Bolzano-Weierstrass theorem. [2]

   (b) Let S be a bounded subset of $\mathbb{R}$ with $SupS = M$ and $InfS = m$. Prove that the set $T = \{x - y : x \in S, y \in S\}$ is a bounded set with $SupT = M - m$ and $InfT = m - M$. [5]

   (c) Prove that $\mathbb{N}$ is unbounded above. [3]

2. (a) Let $S = \{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$. Find S'. [5]

   (b) Is $\mathbb{Q}$ dense in $\mathbb{R}$? Justify your answer. [3]

   (c) Applying L'Hospital rule, find $\underset{x\to 0}{lim} \frac{x - \tan x}{x^3}$. [2]

3. (a) For any $A, B \subset \mathbb{R}$, prove that $(A \cap B)' \subset A' \cap B'$. Is the converse true? Justify . [3]

   (b) State and prove Rolle's theorem. [5]

   (c) Applying LMVT prove that, $0 < log_e(\frac{e^x - 1}{x}) < x,\ x > 0$. [2]

4. (a) If f' exist in $[0, 1]$, show that the equation $2x(f(1) - f(0)) = f'(x)$ has atleast one solution in $(0, 1)$. [3]

   (b) Find a, b, c such that $\underset{x\to 0}{lim} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$. [5]

   (c) If $y = \sin pt, x = \sin t$, prove that $(1 - x^2)y_2 - xy_1 + p^2 y = 0$. [2]

5. (a) If $y = sin(msin^{-1}x)$, prove that

   (i) $(1 - x^2)y_2 - xy_1 + m^2 y = 0$

   (ii) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0$

   [3+3]

   (b) If $f'(x) \geq 0$ on $[1, 2]$, then prove that $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2},\ \forall x, y \in [1, 2]$. [4]

6. (a) By using $(\epsilon - \delta)$ definition, evaluate $\lim\limits_{x\to 2} \frac{2x^2 - 8}{x - 2}$ and find δ if $\epsilon = 0.1$. [3]

   (b) If $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m \neq 0$ then prove that $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{m}$. [5]

   (c) Verify whether $\lim\limits_{x\to 0} \frac{2x+|x|}{2x-|x|}$ exists or not. [2]

7. (a) Examine the continuity of the function

$$f(x) = \begin{cases} \frac{x^4+4x^3+2x}{\sin x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

at $x = 0$. [2]

(b) Let $f : [a, b] \to [c, d]$ and $g : [c, d] \to [e, f]$ be two continuous mappings. Prove that $g \circ f : [a, b] \to [e, f]$ is continuous. [4]

(c) A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x + y) = f(x)f(y)\ \forall x, y \in \mathbb{R}$. If f is continuous at $x = 0$, prove that f is continuous on $\mathbb{R}$. [4]

8. (a) State and prove nested intervals theorem. [5]

(b) Find the limit of the sequence $\{x_n\}$ where $x_n = (2^n + 3^n)^{1/n}$. [3]

(c) Evaluate $\lim_{n\to\infty} \sqrt[n]{n}$. [2]

9. (a) Prove that a sequence is convergent iff it is a Cauchy sequence. [6]

(b) Give an example of a sequence of rational numbers that converges to an irrational number. [1]

(c) Evaluate $\lim_{n\to\infty} \frac{1}{n}\{(2n + 1)(2n + 2)\cdots(2n + n)\}^{1/n}$. [3]

B. A./B. Sc. Examination-2020

Semester-I

Mathematics

CCMA 2 (Algebra I)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any six*** questions.

1. (a) Define zero of a polynomial. Find the multiple root(s) of the polynomial $x^4 - 9x^2 + 4x + 12 = 0$ and solve the equation. [1+4]
   (b) State and prove the factor theorem. [3]
   (c) If $x^2 + px + 1$ is a factor of the polynomial $ax^3 + bx + c$, then show that $a^2 - c^2 = ab$. [2]

2. (a) Let a_1, a_2, a_3, a_4 be the four roots of the equation $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$. Find the value of $(1 + a_1^2)(1 + a_2^2)(1 + a_3^2)(1 + a_4^2)$. [3]
   (b) State the Descarte's rule of signs. Using the Descarte's rule of signs, show that the equation $x^7 + 5x^4 - 3x + k = 0$ can have at most three real roots for every real number k. [1+4]
   (c) Form a biquadratic equation with real coefficients two of whose roots are given by $3i \pm 5$. [2]

3. (a) Let α, β, γ be three roots of the equation $x^3 + qx + r = 0$. Find the equation whose roots are $\frac{1}{\alpha} + \frac{1}{\beta} + 1, \frac{1}{\beta} + \frac{1}{\gamma} + 1, \frac{1}{\gamma} + \frac{1}{\alpha} + 1$. Hence or otherwise find the value of $(\alpha+\beta+\alpha\beta)(\beta+\gamma+\beta\gamma)(\gamma+\alpha+\gamma\alpha)$. [4+1]
   (b) Reduce the cubic equation $8x^3 - 36x^2 + 42x - 5 = 0$ to its standard form and then solve by the Cardan's method. [5]

4. (a) Show that every matrix is row equivalent to a unique matrix in reduced row echelon form. [5]
   (b) Find an equation involving g, h and k that makes the augmented matrix $\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix}$ correspond to a consistent system. Hence or otherwise find $span((1, 0, -2), (-4, 3, 5), (7, -5, -9))$. [3+2]

5. (a) Show that the system

$$\begin{aligned} y + 2z - 3w &= 0 \\ x - y + z - 2w &= 0 \\ -x - 3z + 5w &= 0 \\ x + y + 5z - 8w &= 0 \end{aligned}$$

   has infinitely many solutions. Find a basis and dimension of the solution space. [2+3]
   (b) Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent. Find with justification if the system has unique solution. [3+2]

6. (a) Let A be an $m \times n$ matrix. Prove that the system $Ax = b$ has a solution for every $b \in \mathbb{R}^m$ if and only if $r(A) = m$. [4]
   (b) Let $v_1 = (3, 0, 2)$ and $v_2 = (2, 0, 3)$. Give a geometric description of $span(v_1, v_2)$ without finding it explicitly. [3]
   (c) Let $v_1, v_2, \cdots, v_m$ be m vectors in $\mathbb{R}^n$. Prove that they are linearly dependent if and only if either of them is a linear combination of the others. [3]

7. (a) Let A be a 3×3 matrix with the property that the system $AX = b$ has unique solution for some $b \in \mathbb{R}^3$. Explain why the columns of A span $\mathbb{R}^3$. [3]
   (b) Show that every subspace of $\mathbb{R}^n$ has a basis. [4]
   (c) Find the row space of the matrix $A = \begin{pmatrix} 1 & -2 & 3 \\ 5 & -4 & 3 \\ 9 & -3 & -3 \end{pmatrix}$. Show that the solution space of $AX = 0$ is orthogonal to the row space. [3]

8. (a) Prove that the pivot columns of A form a basis of the column space $C(A)$. [4]

   (b) Suppose that $\mathbb{R}^3 = span(\alpha, \beta, \gamma)$. Explain why $\{\alpha, \beta, \gamma\}$ is a basis of $\mathbb{R}^3$. [3]

   (c) Show that a non null proper subspace of $\mathbb{R}^3$ is either a line through the origin or a plane through the origin. [3]

9. (a) Let A be an $n \times n$ matrix. Prove that the columns of A form a basis of $\mathbb{R}^n$ if and only if A is invertible. [4]

   (b) Find the eigen values of the matrix $\begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ and the eigen vectors belonging to some one of the eigen values. [4]

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   (c) Let A be a square matrix such that $A^2 = 0$. Show that 0 is the only eigen value of A. [2]