B. Sc.(Honours) Examination-2020

Semester-III(CBCS)
Mathematics
Core Course: CC-5
Analysis-III

Time: 3 Hours **Full Marks: 60**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any six*** questions.

1. (a) If f is continuous on $[a, b]$ and $f(a),\ f(b)$ are of opposite signs then show that there is a $\eta \in (a, b)$ such that $f(\eta) = 0$. [5]

   (b) Let $A,\ D \subset \mathbb{R}$ and $f : A \to \mathbb{R}$, $g : D \to \mathbb{R}$ where $f(A) \subset D$. If c is a limit point of A and $\lim\limits_{x \to c} f(x) = l$ then prove the following:
   (i) if $l \in D$ and g is continuous at l then $\lim\limits_{x \to c}(g \circ f)(x) = g(l)$
   (ii) if $l \notin D (l \in D')$ and $\lim\limits_{y \to l} g(y) = m$ then $\lim\limits_{x \to c}(g \circ f)(x) = m$. [5]

2. (a) When a function $f : [a, b] \to \mathbb{R}$ is said to be of bounded variation? If $f,\ g$ be two functions of bounded variation over $[a, b]$ then prove that $f - g$ and fg are of bounded variation on $[a, b]$. [6]

   (b) Verify whether the following functions are of bounded variation or not.
   (i) $f : [0, 1] \to \mathbb{R}$ defined by

$$\begin{aligned} f(x) &= x^2 sin\frac{\pi}{x}, \quad x \neq 0 \\ &= 0, \quad x = 0 \end{aligned}$$

   (ii) $f : [1, 2] \to \mathbb{R}$ defined by $f(x) = 5x^3 - 7x^2 + 12$. [3+1]

3. (a) Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b)$ and $f'(a) \neq f'(b)$. If k be a real number such that $f'(a) < k < f'(b)$ than prove that $\exists c \in (a, b)$ such that $f'(c) = k$. [4]

   (b) If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then show that f is uniformly continuous on $[a, b]$. [4]

   (c) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$. [2]

4. (a) If f and g be two real valued functions, both are bounded and $R-$integrable on $[a, b]$, then prove that fg is also $R-$integrable. Is the converse true? Justify your answer. [5]

   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and f be continuous on $[a, b]$ except on a subset $S \subset [a, b]$ such that the number of limit points of S is finite. Prove that f is $R-$integrable on $[a, b]$. [5]

5. (a) Suppose $f : [a, b] \to \mathbb{R}$ is $R-$integrable on $[a, b]$. Prove that there exist a number η where $inf\{f(x) : a \leq x \leq b\} \leq \eta \leq sup\{f(x) : a \leq x \leq b\}$ such that $(R)\int_a^b f(x)dx = \eta(b - a)$. [5]

   (b) Show that $\dfrac{1}{4} \leq \int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-x^4}} \leq \frac{1}{15}$. [5]

6. (a) Prove that if any one of $(RS)\int_a^b fdg$ and $(RS)\int_a^b gdf$ exists then the other will also exist and $(RS)\int_a^b fdg + (RS)\int_a^b gdf = f(b)g(b) - f(a)g(a)$. [4]

   (b) Let f be continuous and g be such that g' exists, bounded and $R-$integrable on $[a, b]$. Then prove that $(RS)\int_a^b fdg$ and $(R)\int_a^b fg'dx$ exist and are equal. [3]

   (c) Evaluate $(RS)\int_{-\pi}^{\pi}(\pi - |x|)d(|x| + sinx)$. [3]

7. (a) State ratio test for a series of positive terms and give a proof for the case when limit is less than 1. [1+3]

   (b) Give example for the following cases of an infinite series:
   (i) ratio test fails but root test works.
   (ii) ratio test fails but Rabbe's test works. [3+3]

8. (a) State the conditions for convergence of a $p-$series and give the corresponding proof. [1+6]

   (b) How many terms of the following series is required to make the sum correct upto two decimal places: $\sum_{n=1}^{\infty}(-1)^n \, log(1 + \frac{1}{n})$. [3]

9. (a) A real function f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exists at least a point $c \in [0, 1]$ such that $f(c) = f(c + 1)$. [2]

   (b) Show that $|\int_a^b \frac{cosx}{(1+x)}dx < \frac{4}{1+a}$ if $b > a > 0$. [4]

   (c) Verify whether the function $f(x) = \frac{2}{2+x^2}$ is uniformly continuous or not on $\mathbb{R}$. [4]

B. A./B. Sc. Examination-2020

Semesester-III(CBCS)
Mathematics
Core Course: CC-6
(Algebra-II)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***all three*** questions.

1. Answer ***any two*** questions. $(10 \times 2 = 20)$

 (a) i. Let G be a finite group with a normal subgroup N such that $(|N|, |G/N|) = 1$. Then show that N is the unique subgroup of G of order $|N|$. [2]

 ii. If G is a group with center $Z(G)$, and if $G/Z(G)$ is cyclic, then show that G must be abelian. Suppose a noncommutative group G of order 1331 has nontrivial center. Find the order of the center of G. [3+2]

 iii. Let $G =< a >$ be a cyclic group of order 12. Find the order of a^3H in $G/ < a^4 >$. [3]

 (b) i. State and proof Lagrange's Theorem on finite groups. Show that every group of prime order is cyclic. [1+4+2]

 ii. Find all left cosets of $\mathbb{C}^*$ of the subgroup $H = \{z \in \mathbb{C}^* : |z| = 1\}$. Describe them geometrically. [3]

 (c) i. How many permutations are there in S_5 which commute with (1 2 3), and how many of them are even permutations? In S_3, find elements a and b such that $O(a) = 2$, $O(b) = 2$, and $O(ab) = 3$. [3+2+2]

 ii. Show that A_4 is the only subgroup of order 12 in S_4. [3]

2. Answer ***any two*** questions. $(10 \times 2 = 20)$

 (a) i. Let x be a nonzero element of a ring R with unity 1. If there exists a unique $\in R$ such that $xyx = x$, then show that $xy = 1 = yx$. [3]

 ii. Find all units of $M_2(\mathbb{Z})$. [4]

 iii. Construct a non-commutative ring having 81 elements. [3]

 (b) i. Let R be a ring with unity 1. Show that $S = \{n.1|\ n \in \mathbb{Z}\}$ is a subring of R. Give an example of a subring of a ring such that they have different unities. [3+3]

 ii. Show that every finite integral domain is a field. [4]

 (c) Show that characteristic of a finite field is prime. Is every finite ring with prime characteristic a field? Justify your answer. Show that order of every finite field is p^r for some prime p and $r \in \mathbb{N}$. [3+2+5]

3. Answer ***any two*** questions. $(2 \times 10 = 20)$

 (a) i. Let $V = \{(a_1, a_2)|\ a_1, a_2 \in F\}$, where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in V$, define $c(a_1, a_2) = (a_1, 0)$. Is V a vector space over F with these operations? Justify your answer. [4+1]

 ii. Let W be subspaces of a vector space V. Then show there is a subspace W' of V such that $V = W \oplus W'$. Can a vector space be expressed as a union of two its proper subspaces? Justify your answer. [4+1]

(b) i. Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then show that $span(S_1 \cup S_2) = span(S_1) + span(S_2)$. Let $S = \{(1,1,0),(1,0,1),(0,1,1)\}$ be a subset of the vector space $\mathbb{Z}_2^3$. Test the linear independentness of S over $\mathbb{Z}_2$. [3+2]

ii. Show that every finitely generated vector space has a finite basis. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $M_{n\times n}(F)$. Find a basis for W. [3+2]

(c) i. Extend $S = \{(1,0,0,0),(0,2,-1,1)\}$ to a basis of $\mathbb{R}^4$. [4]

ii. Let W be a subspace of a finite dimensional vector space V. Show that $dim(V/W) = dimV - dimW$. [3]

iii. Show that $\mathbb{R}$ is an infinite dimensional vector space over $\mathbb{Q}$. [3]

B. Sc.(H) Examination-2020

Semester-III
Mathematics
Core Course: CC-7
(Differential Equations-I)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *ten* questions from the following: [$10 \times 2 = 20$]
   (a) Solve the initial value problem $(xy^2 - cosx\ sinx)dx + y(x^2 - 1)dy = 0,\ \ y(0) = 2.$ [2]
   (b) Solve the first order differential equation $\frac{dy}{dx} = (y - 5x)^2 + 6,\ \ y(0) = 0.$ [2]
   (c) Solve: $4x^2\frac{d^2y}{dx^2} + y = 0.$ [2]
   (d) Find a third order linear homogeneous differential equation whose linearly independent solutions are x, x^2, and x^3. [2]
   (e) Find the orthogonal trajectories of the family of parabolas $y = cx^2$, where c is a parameter. [2]
   (f) Solve the nonlinear differential equation $\left(\frac{dy}{dx}\right)^2 + (x + y)\frac{dy}{dx} + xy = 0.$ [2]
   (g) Consider the differential equation $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$. If $W(1) = 1$, then find the value of $W(5) - W(2)$. [2]
   (h) Find four different solutions of the differential equation $\frac{dy}{dx} = y^{1/3},\ y(0) = 0.$ [2]
   (i) Solve: $(D^2 + 16)^2(D - 4)^2(D^2 - 16)^2y = 0.$ [2]
   (j) Find an integrating factor of the differential equation $y(ydx - xdy) + x^2(2ydx + 2xdy) = 0.$ [2]
   (k) Find the differential equations of all circles passing through the origin and having their centers on the axis of x. [2]
   (l) Let $x(t)$ be the solution of the differential equation $\frac{dx}{dt} = (x - 3)(x - 5)(x - 11)(x - 15)$ satisfying the initial condition $x(0) = 6$. Find the value of $x(t)$ when $t \to \infty$. [2]
   (m) Radium disappears at a rate proportional to the amount present. If 10% of the original amount disappears in 100 years, how much will remain present at the end of 200 years. [2]
   (n) Explain why it is always possible to express any homogeneous differential equation $M(x, y)dx + N(x, y)dy = 0$ in the form $\frac{dy}{dx} = F\left(\frac{x}{y}\right)$. [2]

2. (a) Prove that the transformation $v = y^{1-n}(n \neq 0\ or\ 1)$ reduces the Bernoulli equation $\frac{dy}{dx} = P(x)y + Q(x)y^n$ to a linear equation in v. Hence solve the equation $x\frac{dy}{dx} + y = y^2x^3cosx$. [2+2]
   (b) Solve the second order linear differential equation $\frac{d^2y}{dx^2} - y = xsinx + (1 + x^2)e^x$. [4]
   (c) Solve: $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 9 = 0.$ [2]

3. (a) Solve the nonlinear differential equation $y = 2x\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^4$. [3]

(b) Solve the equation $x(1+x)\frac{d^2y}{dx^2} + (2-x^2)\frac{dy}{dx} - (2+x)y = x(1+x)^2$, where $y = e^x$ and $y = \frac{1}{x}$ are two linearly independent solutions of the corresponding homogeneous equation. [4]

(c) If u and v are any two solutions of the equation $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$ on an interval $[a, b]$, then prove that their Wronskian $W(u, v)$ is either identically zero or never zero on $[a, b]$. [3]

4. (a) Solve the homogeneous differential equation $(x^2 - y^2)dx + 2xydy = 0$. [3]

(b) Let x^m is an integrating factor of the differential equation $2x^2(x+1)\frac{d^2y}{dx^2} + x(7x+3)\frac{dy}{dx} - 3y = x^2$. Find the possible values of m. [3]

(c) Find $v(x)$ such that $y(x) = e^{4x}v$ is a particular solution of the differential equation $(D^2 - 8D + 16)y = (2x + 9x^8 + 13x^{12})e^{4x}$. Hence find the general solution of it. [4]

5. (a) Using the method of undetermined coefficients, solve the equation $(D^2 + D - 6)y = 10e^{2x} - 18e^{3x} - 6x - 11$. [3]

(b) Find the singular solution and extraneous loci of the differential equation $4p^2x = (3x-1)^2$. [4]

(c) Solve the equation $(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + 4y$ by changing the independent variable. [3]

6. (a) Reduce the equation $(x^2+y^2)(1+p)^2 - 2(x+y)(1+p)(x+py) + (x+py)^2 = 0$ to Clairaut's form by the substitution $x^2 + y^2 = v$ and $x + y = u$. Hence solve the equaion. [4]

(b) Solve the differential equation $(x+2)\frac{dy}{dx} + y = f(x)$ with initial condition $y(0) = 4$, where $f(x) = 2x$, when $0 \leq x < 2$ and $f(x) = 4$, when $x \geq 2$. Hence find the value of $y(5)$. [5]

(c) Use the method of isoclines to sketch some of the solution curves of the equation $\frac{dy}{dx} = e^{-x^2}$. [1]

7. (a) If $e^{\int \phi(\frac{y}{x})d(\frac{y}{x})}$ is an integrating factor of the differential equation $M(x,y)dx + N(x,y)dy = 0$, then find the expression of $\phi(\frac{y}{x})$. [3]

(b) Solve the differential equation $(xy^2 - e^{\frac{1}{x^3}})dx - (x^2y)dy = 0$. [4]

(c) Find the equation of the family of curves which cut the members of the family of hyperbolas $y^2 + 2xy - x^2 = c$ at angle of 45^o. [3]

B. Sc. (Honours) Examination-2020

Semesester-III(CBCS)

Mathematics

Course: SEC-1

(Boolean Algebra and Circuit Design)

Time: Three Hours **Full Marks: 25**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any five*** questions.

1. In a Boolean algebra B, show that $a + (a.b) = a$ for every $a, b \in B$. Given an example of a Boolean algebra. [3+2]

2. Let $(R, +, \cdot)$ be a Boolean ring. Define binary operations $\vee, \wedge$ and a unary operation $'$ on R by
$$a \vee b = a + b + a \cdot b; a \wedge b = a \cdot b; a' = I + a,$$
I being the unit element in R, then shoe that $(R, \vee, \wedge, ')$ is a Boolean algebra. [5]

3. Show that every Boolean algebra is atomic. [5]

4. Write the expression in disjunctive normal form $(x + y)(y + z)(x' + y' + z')$. Define Boolean function of n variables. [3+2]

5. Represent the Boolean function $f(x, u, v, w) = \sum(0, 1, 4, 5, 9, 11)$ in a Karnaugh map. Write down the minimal form of f. [5]

6. Define 'And' gate. Draw the circuit which realises the Boolean expression $(A + B')(C + D' + E)(A + C' + E')$ using 'OR-AND' gate. [5]

7. Draw the circuit which realises the Boolean expression $(x + y + z')(x' + y + z)(x + y' + z)$. Convert the number $(139)_{10}$ into binary number. [3+2]

8. Add and subtract the numbers $(10111)_2$ and $(1001101)_2$. [2+3]