M. Sc. Examination-2020

Semester-I

Mathematics

Course : MMC-11

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer question no. 6 and any **three** from the rest

1. (a) Define Lebesgue measure of a set. Show that every open set is measurable. [1+5]

   (b) Show that the class of all Lebesgue measurable sets is closed under countable union and countable intersection . [6]

2. (a) Define mesurability of a function. Show that continuous functions, monotone functions, and simple functions are measurable. [$1+1\frac{1}{2}+1\frac{1}{2}+2$]

   (b) State and prove Lusin theorem [6]

3. (a) Show that the following are equivalent.
   i) E is measurable.
   ii) $\forall \epsilon > 0$ $\exists$ an open set O such that $E \subset O$ and $\mu^*(O - E) < \epsilon$.
   iii) $\exists$ a G_δ set G such that $E \subset G$ and $\mu^*(G - E) = 0$. [1+5]

   (b) If f is measurable then show that $|f|$ is also measurable . [6]

4. (a) Define Lebesgue integral of a bounded function. Show that every bounded measurable function is Lebesgue integrable. [2+4]

   (b) Show that the class of all bounded Lebesgue integrable functions forms a linear space. [6]

5. (a) State and prove Bounded convergence theorem. Show that it is not true for Reimann integration. [1+4+3]

   (b) Define summable function. State Monotone convergence theorem. [2+2]

6. Answer any two

   (a) If E_1 and E_2 are measurable sets such that $E_1, E_2 \subset [0, 1]$, and $\mu(E_1) = \mu(E_2) = 1$ then show that $\mu(E_1 \cup E_2) = 1 = \mu(E_1 \cap E_2)$. [2]

   (b) Let $f(x) = \frac{1}{\sqrt{x}}$ for $x > 0$ and let $f(0) = 0$. Prove that f is summable on $[0, 1]$ [2]

   (c) Show that Cantor set is measurable and has measure zero [2]

M. Sc. Examination - 2020

Semester-I

Mathematics

Paper: MMC 12 (New)

(Complex Analysis)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Let $f(z)$ be a continuous function in a simply connected domain D and α be any arbitrary but fixed point in D. If the integral $g(z) = \int_{\alpha}^{z} f(w)dw$ is independent of the path so long as the path lies in D, then show that $g'(z) = f(z)$ for every z in D. [4]

   (b) If $f(z)$ be a non-constant analytic function regular in a domain D including the boundary C and $f(z) \neq 0$ for any point in D, then show that $|f(z)|$ can not attain its minimum in D. [3]

   (c) Show that $f(z) = z^{10} - z^4 + 1$ vanishes at least for one point in $|z| < 2$. [3]

2. (a) State Laurent's theorem. [2]

   (b) If $f(z) = e^{\frac{1}{2}k(z - \frac{1}{z})}$, where k is a constant, then show that $f(z)$ can be expanded in the form $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - k\sin\theta)d\theta$. [3]

   (c) Define a 'removable singularity'. If a function f is bounded and analytic throughout a domain $0 < |z - \alpha| < \delta$, then prove that either f is analytic at α or else α is a removable singularity of f. [1+4]

3. (a) Find the singularities of the function $f(z) = \sin\frac{1}{z-2} + \sin\frac{1}{z+2}$ and classify the singularities. [3]

   (b) If $f(z)$ be a non-constant function regular in a region R, then prove that $f(z)$ have at most a finite number of zeros in every closed sub region of R. [3]

   (c) Define a 'zero' of an analytic function. Prove that the zeros of a non-constant analytic function are isolated points. [1+3]

4. (a) If $f(z)$ is analytic in $|z| < 1, f(0) = 0$ and $|f(z)| \leq 1$, then show that $|f(z)| \leq |z|$. [2]

   (b) State and prove Cauchy's residue theorem. Using this theorem evaluate $\oint_{|z|=1} \frac{e^{3z}}{(4z-\pi)^2}dz$. [5]

   (c) If α be a pole of $f(z)$ of order p, then show that the residue of $f(z)$ at α is given by $\frac{1}{(p-1)!}\lim_{z\to\alpha}\frac{d^{p-1}}{dz^{p-1}}[(z-\alpha)^p f(z)]$. [3]

5. (a) State and prove Rouche's theorem(Principle of Argument may be assumed). [4]

   (b) Show that the roots of the equation $16z^5 - z + 8 = 0$ all lie in the annulus between $|z| = \frac{1}{2}$ and $|z| = 1$. [4]

   (c) Evaluate the integral $\oint_{|z|=\frac{5}{2}} \frac{f'(z)}{f(z)}dz$, where $f(z) = \frac{(z^2+1)^2}{(z^2-3z+2)^3}$. [2]

6. (a) Prove that a function which is analytic every where including the point at infinity is a constant. [3]

   (b) Find $Res[f(z), \infty]$, where $f(z) = \frac{z^2}{z^2-1}$. [2]

   (c) Evaluate any one of the following by the method of contour integration: (i)$\int_{-\infty}^{\infty} \frac{x\sin x dx}{a^2x^2+b^2}, a < 0 < b$; (ii)$\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta}, |a| < 1$. [5]

M. Sc. Examination-2020

Semester-I

Mathematics

MMC-13(New)(Linear Algebra)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.

(V is a finite dimensional vector space over F.)

Answer ***any four*** questions.

1. (a) Define T-cyclic subspace. Consider the linear operator $D : P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ defined by $D(p(x)) = p'(x)$. Find the D-cyclic subspace generated by $x^2 + x + 1$. [1+2]

   (b) Let V be a two dimensional vector space and $T : V \longrightarrow V$ be a linear operator. Show that either V is T-cyclic or $T = cI$ for some scalar c. [3]

   (c) Let $T : V \longrightarrow V$ be a linear operator such that the characteristic polynomial splits over F and has the distinct eigen values $\lambda_1, \lambda_2, \cdots, \lambda_k$. Show that T is diagonalizable if and only if $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$. [4]

2. (a) Let $T : V \longrightarrow V$ be a linear operator. Then there is a unique monic nonconstant polynomial $m(t)$ of least degree such that $m(T) = T_0$. [3]

   (b) Let $T : V \longrightarrow V$ be a linear operator and $m(t)$ is the minimal polynomial of T. Prove that $\lambda \in F$ is an eigen value of T if and only if $m(\lambda) = 0$. [2]

   (c) Show that the linear operator $T(x, y, z) = (x - 3y + 3z, 3x - 5y + 3z, 6x - 6y + 4z)$ is diagonalizable. Find a basis β of $\mathbb{R}^3$ such that $[T]_\beta$ is a diagonal matrix. [3]

   (d) Let A be an $n \times n$ complex matrix such that every nonzero vector in $\mathbb{C}^n$ is an eigen vector of A. Is A a scalar matrix? [2]

3. (a) Let λ be an eigen value of a linear operator T and G_λ be the corresponding generalized eigen space. Show that the subspace G_λ is T-invariant and λ is the only eigen value of $T|_{G_\lambda}$. [3]

   (b) Let $T : V \longrightarrow V$ be a linear operator. Prove that nonzero generalized eigen vectors corresponding to distinct eigenvalues of T are linearly independent. [3]

   (c) Let T be a linear operator on V whose characteristic polynomial splits over F, and λ be an eigenvalue of T. If a basis for G_λ is the union of q disjoint cycles of generalized eigenvectors, prove that $q \geqslant \dim E_\lambda$. When does the equality hold? Justify. [2+2]

4. (a) Let $T : V \longrightarrow V$ be a nilpotent linear operator. Prove that k is the index of nilpotence of T if and only if t^k is the minimal polynomial of T. [3]

   (b) Let $T : V \longrightarrow V$ be a nilpotent operator. Prove that the number of Jordan blocks in a Jordan canonical form is equal to the geometric multiplicity of the eigen value 0. [3]

   (c) Let T be a linear operator which has the characteristic polynomial $t^2(t-1)^4$ and the minimal polynomial $t(t-1)^2$. Find all possible Jordan canonical forms of T. In each case find the rank and nullity of the matrix; and the dimension of the eigen spaces. [4]

5. (a) Consider the inner product space $P_2(\mathbb{R})$ where the inner product is defined by $< p(x), q(x) >= \int_0^1 p(x)q(x)dx$ and a linear functional ϕ on $P_2(\mathbb{R})$ is defined by $\phi(a + bx + cx^2) = a + 2b + 3c$. Find an orthonormal basis of this inner product space and find $q(x) \in P_2(\mathbb{R})$ such that $\phi(p(x)) =< p(x), q(x) >$. Also find the adjoint of the linear operator $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by $\phi(p(x)) = p(0) + xp'(x)$. [3+1+2]

   (b) Let $T : U \longrightarrow V$ be a linear transformation. Show that $ker\ T^* = (Im\ T)^\perp$ and $ker\ TT^* = ker\ T^*$. [2+2]

6. (a) Let T be a normal operator on an Euclidean space V. Show that the eigen vectors belonging to two distinct eigen values are orthogonal and the minimal polynomial of T is a product of distinct irreducible factors over $\mathbb{R}$. [1+2]

   (b) Let T be an unitary operator over a complex inner product space V. Prove that T maps every orthonormal basis of V into an orthonormal basis of V. [3]

   (c) Let T be a linear operator over an Euclidean space V. Show that T is symmetric if and only if it is orthogonally diagonalizable. [4]

MSc Examination: 2020
Semester-I
Mathematics
Paper- MMC-14
(Ordinary differential equations)

Time: Three hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four (04)*** **questions out of six(06).**

1. (a) Find the characteristic values and corresponding orthonormal characteristic functions of following ordinary differential equation

$$\frac{d^2x}{dy^2} + \lambda x = 0.$$

with boundary conditions

$$(i) x(0) = 0, \quad x(\pi) = 0 \quad and$$
$$(ii) x'(0) = 0, \quad x(\pi/2) = 0$$

[3+3]

(b) Using Green's function, solve the following differential equation

$$\frac{d^2x}{dy^2} + \omega^2 x = f(y).$$

with boundary conditions

$$x(0) = 0, \quad x(\pi/2) = 0$$

[4]

2. (a) State an existence and uniqueness theorem for solution of a 1st order IVP. With a suitable example show the theorem is sufficient but not necessary. [1+2]

(b) Show that the interval where the unique solution of the following IVP

$$\frac{dy}{dx} = \frac{e^{y^2} - 1}{1 - x^2y^2}.$$

subject to the condition

$$y(-2) = 1,$$

surely exist is [-2.02, -2.18] [4]

(c) Use Picard's iteration method to find the first two approximate solution of the following IVP
$\frac{dx}{dt} = cosx,$ subject to $x(0) = 0.$
Also find how good the approximation is. [2+1]

3. (a) State Abel's formula. Hence show that if f and g are two solutions of

$$\frac{d}{dt}(P(t)\frac{dx}{dt}) + Q(t)x = 0,$$

such that f and g have a common zero then f and g are linearly dependent. [1+3]

(b) Find the interval of definition of the following IVP
$\frac{d^3x}{dt^3} + \frac{x}{t^2-4} = cost$, subject to $x'(1) = 3 \quad x'(1) = 0 \quad x''(1) = 0$. [3]

(c) Show that $\{1, \quad t, \quad t^2\}$ are linearly independent over $(-\infty, +\infty)$.
Construct two functions f_1 and f_2 such that they are linearly independent but their Wronskian is zero over $(-\infty, +\infty)$. [1+2]

4. (a) What is meant by a fundamental matrix of a linear homogeneous system

$$\frac{dX}{dt} = AX.$$

Express the coefficient matrix A in terms of fundamental matrix.

If

$$\Phi(t) = \begin{bmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{bmatrix}$$

is a fundamental matrix of $\frac{dX}{dt} = AX.$, then find A. [1+2+3]

(b) Solve the following system

$$\begin{aligned} \frac{dx}{dt} &= x + y - 1 \\ \frac{dy}{dt} &= y + 1 \end{aligned}$$

subject to the initial condition $x(0) = 1$ and $y(0) = 0$. [4]

5. (a) Find the nature and stability of the critical point of the following system
$x' = 3x - 4y, \quad y' = x - y.$
Also find and sketch the phase path. [3+2]

(b) Define matrix exponential function e^{At}.
If $P^{-1}AP = D$ then show that $e^{At} = Pe^{Dt}P^{-1}$. [2+3]

6. (a) State and prove Sturm separation theorem. [1+4]

(b) Convert the Bessel's equation of order n into the form $x' + p(t)x = 0$.
Show that if $0 \leq n < \frac{1}{2}$ the distance between two successive zeros of $x(t)$ is less than π and tends to π as the number of oscillating solution increases. [2+3]

M. Sc. Examination-2020

Semester-I

Mathematics

Paper: MMC 15

(Partial Differential Equations)

Time: 03 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Show that the Cauchy problem $yu_x - xu_y = 0$, $u = f$ on Γ has a unique solution in a neighbourhood of Γ for every differentiable function $f : \Gamma \to \mathbb{R}$, where $\Gamma = \{(x, y) : x + y = 1, x > 1\}$.
(b) In what region is the following PDE parabolic, elliptic or hyperbolic?

$$\frac{(x-y)^2}{4}u_{xx} + (x-y)cos(x^2+y^2)u_{xy} + sin^2(x^2+y^2)u_{yy} + (x-y)u_x + cos^2(x^2+y^2)u_y + u = 0$$

(c) Find, by the method of characteristics, the integral surface of the following PDE:

$$\frac{\partial u}{\partial y} + (\frac{\partial u}{\partial x})^2 = 1,\ x \in \mathbb{R}, y > 0,\ u(x,0) = -x^2.$$

[2+3+5]

2. (a) Find the complete integral of the PDE

$$u_{xx} - u_{xy} - 2u_{yy} + 2u_x + 2u_y = xy + sin(x+2y).$$

(b) Solve the following PDE by Monge's method:

$$y^2\frac{\partial^2 u}{\partial x^2} - 2y\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = u_x + 6y.$$

[4+6]

3. (a) Find the expression of the Monge cone of the PDE $u_x^2 + u_y^2 = 2021$ at $(1, 1, 1)$.
(b) Find the number of real characteristic curves of the PDE

$$[a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}][c(x,y)\frac{\partial}{\partial x} + d(x,y)\frac{\partial}{\partial y}]u = 0,\ ad - bc = 0,$$

where a, b, c, d be four differentiable functions defined on $\mathbb{R}^2$.
(c) Find the two families of characteristics of

$$2x^2u_{xx} - 5xyu_{xy} + 2y^2u_{yy} + 2xu_x + 2yu_y = 0.$$

Convert the equation to canonical form and hence find $u(x, y)$, given that $u = 2x^2 - 6$ and $u_y = 3 - x^2$ on $y = 1$. [2+2+6]

4. (a) Using appropriate integral transform method, solve the initial boundary value problem (IBVP) described as

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - cos(2t),\ 0 \le x < \infty, 0 \le t < \infty,$$

BC's: $u(0,t) = 0$, $u(x,t)$ is bounded as $x \to \infty$,
IC's: $u(x,0) = 0$, $u_t(x,0) = 0$.
(b) Using appropriate integral transform method, find the temperature $u(x,t)$ in a semi-infinite rod $0 \le x < \infty$, determined by the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ x \in (0,\infty), t > 0,$$

subject to
IC: $u(x,0) = 0,\ 0 \le x < \infty$,
BC's: $u_x(0,t) = -u_0$ (a constant) when $x = 0$ and $t > 0$;
u, $\frac{\partial u}{\partial x}$ both tend to zero as $x \to \infty$. [5+5]

5. (a) State maximum-minimum principle for a function $u(x, y)$ which is continuous in a closed region $\overline{\mathbb{R}}$ and satisfies the Laplace equation $\nabla^2 u = 0$ in the interior of $\mathbb{R}$.
(b) Let $u(x, t)$ satisfies the initial boundary value problem (IBVP)

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}, \; x \in (0, 1), t > 0,$$

BC's: $u(0, t) = u(1, t) = 0$,
IC: $u(x, 0) = sin^2(\pi x), \; x \in [0, 1]$.

Then, for $x \in (0, 1)$ find the value of $u(x, \frac{1}{2\pi^2})$.
(c) With the help of separation of variables method, obtain the solution of the following initial boundary value problem (IBVP) for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \; x \in (0, \pi), t > 0,$$

BC's: $u(0, t) = u(\pi, t) = 0$,
IC's: $u(x, 0) = 0, \; u_t(x, 0) = 2cos(x) - 3cos(2x)$. [2+3+5]

6. (a) State Cauchy-Kowalevski theorem for a second order linear PDE.
(b) Solve the following interior Dirichlet problem for a circle:
PDE: $\nabla^2 u = 0$, $0 \leq r < a$, $0 \leq \theta < 2\pi$
BC: $u(a, \theta) = f(\theta)$, $0 \leq \theta < 2\pi$
where $f(\theta)$ is a continuous function on $\partial\mathbb{R}$.
Hence deduce Poisson's integral formula of the form
$u(r, \theta) = \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \frac{(a^2 - r^2) f(\phi)}{a^2 - 2arcos(\phi - \theta) + r^2} \, d\phi$. [2+8]

M. Sc. Examination-2020

Semester-I

Mathematics

Paper: MMC-16

(Integral Transforms and Integral Equations)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.

Notations and symbols have their usual meanings.

Answer Question No. 1 and any *three* from the rest.

1. Answer any *five* questions from the following: [$5 \times 2 = 10$]

   (a) Solve the Fredholm integro-differential equation
   $u'(x) = -1 + cosx + \int_0^{\frac{\pi}{2}} tu(t)dt,\ u(0) = 0.$ [2]

   (b) Find the Laplace transform of $f(t)$, where $f(t) = e^{-2t}t^3 + e^{-7t}sin5t - tcos3t$. [2]

   (c) Find the value of $u(0) + u(\frac{\pi}{4}) + u(\frac{\pi}{2})$, where $u(x)$ satisfies the equation
   $\frac{d^2u}{dx^2} = -x + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u'(0) = 1.$ [2]

   (d) Using modified Adomian decomposition method, solve the integral equation
   $y(x) = cosx - x(1 - e^{sinx}) - \int_0^x xe^{sint}y(t)dt.$ [2]

   (e) Convert the integral equation $y(x) = 5x^2 + \int_0^1 k(x,t)y(t)dt$ to an equivalent differential equation, where
   $k(x,t) = t(1-x)$ when $0 \le t \le x$ and $k(x,t) = x(1-t)$ when $x \le t \le 1$. [2]

   (f) Find the inverse Laplace transform of $F(s)$, where $F(s) = \frac{s}{s^2+4s+7} + \frac{1}{(s+3)^4}$. [2]

   (g) Find the eigenvalues and eigenfunctions of the integral equation
   $y(x) = \lambda \int_0^5 e^x ty(t)dt.$
   [2]

2. (a) Find the resolvent kernel of the Fredholm integral equation
   $y(x) = x^{11} + \lambda \int_0^{2\pi} sin(x+t)y(t)dt.$ [4]

   (b) Convert the differential equation $\frac{d^2y}{dx^2} + 4xy = 2, y(0) = 0, y'(1) = 1$ to an equivalent integral equation. [3]

   (c) Find the inverse Fourier sine transform of $\frac{e^{-as}}{s}$. Hence find the value of $\int_0^\infty \frac{sin2x}{x}dx$. [2+1]

3. (a) Using Hilbert-Schmidt theorem, solve the integral equation $y(x) = x + \int_0^1 (1-3xt)y(t)dt$. [4]

   (b) If f and g are piecewise continuous on $[0,\infty)$ and of exponential order, then prove that $L\{f * g\} = L\{f\}.L\{g\}$.
   Using the above result prove that $\beta(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$. [2+1]

   (c) If $L\{f(t)\} = F(s)$, then prove that $L\{\frac{f(t)}{t}\} = \int_s^\infty F(x)dx$. Hence evaluate the integral $\int_0^\infty \frac{e^{-4t}sint}{t}dt$. Does the Laplace transform of $\frac{cosat}{t}$ exist? [1+1+1]

4. (a) State Fredholm alternative theorem. Find the conditions for which the integral equation $y(x) = f(x) + \int_0^1 k(x,t)y(t)dt$ has infinitely many solutions, where
   $k(x,t) = (t+1)x$ when $0 \le t \le x$ and $k(x,t) = (x+1)t$ when $x \le t \le 1$. [1+3]

   (b) Using Laplace transform, solve the differential equation
   $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y = h(t), y(0) = 0, y'(0) = 1,$
   where $h(t) = 3$ when $0 \le t \le \frac{\pi}{2}$ and $h(t) = 0$ when $t > \frac{\pi}{2}$. [3]

(c) Solve the differential equation $t\frac{dx}{dt} - x = 1$ by Laplace transform method. [2]

(d) Give an example of a function which is neither piecewise continuous nor of exponential order but whose Laplace transform exists. [1]

5. Solve the Volterra integral equation $y(x) = x - \int_0^x (x-t)y(t)dt$ by

(a) Laplace transform method. [2]

(b) series solution method. [2]

(c) Adomian decomposition method. [2]

(d) successive approximations method. [2]

(e) successive substitutions method. [2]

6. (a) Solve the Volterra integral equation of first kind $e^{x^2/2} - 1 = \int_0^x sin(x-t)u(t)dt$. [2]

(b) Find the Fourier transform of $f(x)$,
where $f(x) = 1 - x^2$ if $|x| < 1$ and $f(x) = 0$ if $|x| > 1$.
Hence using Persaval's identity, evaluate $\int_0^\infty (\frac{sinx - xcosx}{x^3})^2 dx$. [3+1]

(c) Prove that $L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-Ts}}$, where $f(t)$ is a periodic function with period T. Hence find the Laplace transform of the following periodic function [2+2]