B. Sc. (Honours) Examination-2021

Semester-III (CBCS)

Mathematics

Course: CCMA-5

(Analysis III)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.

Notations and symbols have their usual meanings.

Answer ***any six*** questions.

1. (a) Examine whether the series $\sum_{n=1}^{\infty} \frac{\sin n\alpha}{e^n}$ $(\alpha > 0)$ is convergent. [3]

   (b) Prove or disprove: If $\sum a_n$ converges, then $\sum a_{3n}$ converges too. [4]

   (c) Let $\{a_n\}$ be a monotone decreasing sequence of non-negative real numbers such that $na_n \to 0$ as $n \to 0$. Discuss the convergence of the series $\sum a_n$. [4]

2. (a) Examine the convergence of the series $1 + \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \cdots$. [4]

   (b) Find all suitable values of b_n $(n \geq 1)$ such that $\sum a_n b_n$ converges if and only if $\sum a_n$ converges absolutely. [4]

   (c) Prove or disprove: If $\sum a_n^2$ diverges, then $\sum a_n$ also diverges. [2]

3. (a) Let $f : I \longrightarrow \mathbb{R}$ be a nonconstant function on an interval I. Is $f(I)$ necessarily an interval? Justify your answer. [4]

   (b) Construct a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous on $\mathbb{R} \setminus \{1, 2, \ldots\}$. [3]

   (c) Show by examples that a real-valued continuous function on an interval may or may not be bounded. [3]

4. (a) Discuss the difference of the continuity and uniform continuity of a function on some interval $[a, b]$. Determine whether the function $f : [0, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$. [2+4]

   (b) If a function $f : [a, b] \longrightarrow \mathbb{R}$ possesses the intermediate value property on $[a, b]$ and $f(x) \neq f(y)$ for any two distinct points $x, y \in [a, b]$, then show that f is uniformly continuous on $[a, b]$. [4]

5. (a) What are the total variation and the variation function of a function of bounded variation on a closed and bounded interval? Show that the variation function V of a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$. [2+4]

   (b) Let $I = [a, b]$. Then prove or disprove: A uniformly continuous function on I is a function of bounded variation on I. [4]

6. (a) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function on $[a, b]$. Define the Riemann integral of f in two different approaches. [4]

   (b) If $f : [a, b] \longrightarrow \mathbb{R}$ is a bounded function on $[a, b]$ such that f^2 is Riemann integrable on $[a, b]$, then is f necessarily Riemann integrable on $[a, b]$? Justify. [3]

   (c) Examine whether the function $f : [0, 1] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3n + 2 & \text{for} \quad x = \frac{1}{n} \quad \text{(where } n = 1, 2, \ldots) \\ 0 & \text{otherwise} \end{cases}$$

   is Riemann integrable on $[0, 1]$. [3]

7. (a) If $f : [a,b] \longrightarrow \mathbb{R}$ be Riemann integrable function on $[a,b]$, then show that the function $g : [a,b] \longrightarrow \mathbb{R}$ defined by $g(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$. [4]

(b) State the Fundamental Theorem of Integral Calculus. Examine whether we can apply it to evaluate the integral $\int_1^3 f dx$ of the function $f : [1,3] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3 & \text{for} \quad 1 \leq x < 2 \\ 5 & \text{for} \quad 2 \leq x \leq 3 \end{cases}$$

[2+4]

8. (a) What do you mean by '$f \in \mathcal{R}(\alpha)$ on $[a,b]$' for suitable functions f and α? If $\alpha : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ and $f : [a,b] \longrightarrow \mathbb{R}$ is monotone increasing on $[a,b]$, then show that $\int_a^b \alpha df = \alpha(c)[f(b) - f(a)]$ for some $c \in [a,b]$. [2+4]

(b) Examine whether $x - [x] \in \mathcal{R}(\sin x)$ on $[0, \frac{\pi}{2}]$ and evaluate $\int_0^{\frac{\pi}{2}} (x - [x]) d(\sin x)$, if it exists. [4]

B. Sc. Examination-2021

Semesester-III(CBCS)
Mathematics
Core Course: CC-6
(Algebra-III)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***all three*** questions.

1. Answer ***any two*** questions. $(10 \times 2 = 20)$
   (a) i. Show that the elements in A_n are product of 3-cycles. Let $\beta \in S_7$ and suppose $\beta^4 = (2143567)$. Find β. [3+2]
   ii. Find group elements α and β in S_5 such that $|\alpha| = 3$, $|\beta| = 3$, and $|\alpha\beta| = 5$. [3]
   iii. Show that in $S_n(n \geq 4)$, the equation $x^2 = (1234)$ has no solutions.
   (b) i. Describe and discuss the natures of all subgroups of the noncommutative group Q_8. Are there any similarities with the subgroups of an abelian group? Justify [4+1]
   ii. Let G be a group and $[G : Z(G)] = 4$, then find the orders of the elements of the group $G/Z(G)$. Can you find any similarities with the Kelin's 4-group? Justify [2+1]
   iii. Show, by example, that in a quotient group G/H it can happen that $aH = bH$ but $O(a) \neq O(b)$.
   (c) i. Let G be a finite group. Then prove that $a^{|G|} = e$ for all $a \in G$. Suppose that K is a proper subgroup of H and H is a proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of H? [3+2]
   ii. Let H be a subgroup of R^*, the group of nonzero real numbers under multiplication. If $R^+ \subseteq H \subseteq R^*$, prove that either $H = R^+$ or $H = R^*$. [3]
   iii. Prove that the order of the group U_n is even when $n > 2$. [2]

2. Answer ***any two*** questions. $(10 \times 2 = 20)$
   (a) Find the group of all units in $\mathbb{Z}[i]$. [2]
   (b) Show that every Boolean ring is commutative. Give example of a nonzero Boolean ring other than $\mathbb{Z}_2$ and products of $\mathbb{Z}_2$. [3]
   (c) Let R be a commutative ring. If $u \in R$ is a unit and $a \in R$ is a nilpotent element, then show that $u + a$ is a unit in R. [3]
   (d) Let R be a ring such that $x^3 = x$ for all $x \in R$. Show that the characteristic of R is finite and is a divisor of 6. [2]

3. (a) Give an example with justification of a division ring which is not a field. [2]
   (b) Prove that the characteristic of a finite field is a prime integer. [3]
   (c) Show that there is no integral domains of six elements. [3]
   (d) Let R be a ring with 1 such that any two nonzero elements in R is nonzero. Show that for every $a, b \in R$, $ab = 1$ if and only if $ba = 1$. [2]

4. (a) Find the smallest subring of $\mathbb{R}$ containing $\sqrt[3]{2}$. [2]
   (b) Let R be a commutative ring with 1. If R is simple then show that it is a field. Give an example of a simple noncommutative ring with 1. [3]

(c) Show that $12\mathbb{Z} + 28\mathbb{Z} = 4\mathbb{Z}$. [2]

(d) Show that $I = \{a + ib \in \mathbb{Z}[i] \mid a \text{ and } b \text{ are even}\}$ is an ideal of the ring $\mathbb{Z}[i]$ of Gaussian integers. Find the number of elements of the quotient ring $\mathbb{Z}[i]/I$. [3]

5. Answer ***any two*** questions. $(2 \times 10 = 20)$

(a) i. Show that every field is a vector space over a subfield of it. Give an example with proper justification to show that a subfield may not be a vector space over its field. [4+1]

ii. Show that the set $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$ forms a subspace of $\mathbb{R}^3$. Hence find a subspace W' of $\mathbb{R}^3$ such that $\mathbb{R}^3 = W \oplus W'$. [2+3]

(b) i. Let $W = \{A \in M_{2\times 2}(\mathbb{R}) : \text{Trace}(A) = 0\}$. Find a smallest generating set for W. [3]

ii. Give an example of a linearly independent set having more than one element for the vector space $P(\{1, 2, 3\})$ over the field $\mathbb{Z}_2$. [3]

iii. Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then show that $S \cup \{v\}$ is linearly dependent if and only if $v \in span(S)$. [4]

(c) i. Let V be finite dimensional vector space over F. Then show that every linearly independent subset of V can be extended to a basis of V. [4]

ii. Find the dimension of the subspace $W_1 + W_2$ of $\mathbb{R}^3$ where $W_1 = \{(a, b, c) : a + b + c = 0\}$ and $W_2 = \{(a, b, c) : a = b = c\}$ are two subspaces of $\mathbb{R}^3$. [6]

B. Sc.(H) Examination-2021

Semester-III

Mathematics

Core Course: CC-7

(Differential Equations-I)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *ten* questions from the following: [$10 \times 2 = 20$]
   (a) Solve the differential equation $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$. [2]
   (b) Solve: $(D^2+9)^2(D^2+3D+2)^2(D+2)^3y = 0$. [2]
   (c) Solve: $\frac{dy}{dx} = \frac{7xy+y^2}{7x^2}$. [2]
   (d) Find a second order linear homogeneous differential equation whose linearly independent solutions are x, and xe^x. [2]
   (e) Find the orthogonal trajectories of the family of parabolas $cy^2 = x^3$. [2]
   (f) Solve the nonlinear differential equation $sin(x\frac{dy}{dx})cos\ y = cos(x\frac{dy}{dx})sin\ y + \frac{dy}{dx}$. [2]
   (g) Find four different solutions of the differential equation $\frac{dy}{dx} = y^{2/3}$, $y(0) = 0$. [2]
   (h) Solve: $y\frac{d^2y}{dx^2} = (\frac{dy}{dx})^2$. [2]
   (i) Let $y(x)$ be the solution of the differential equation $\frac{dy}{dx} = (y-11)(y-19)(y-25)$ satisfying the initial condition $y(0) = 15$. Find the value of $y(x)$ when $x \to \infty$. [2]
   (j) Consider the differential equation $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$. If $W(1) = 1$, then find the value of $W(7) - W(3)$. [2]
   (k) Find an integrating factor of the differential equation $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$. [2]
   (l) Explain why it is always possible to solve any homogeneous differential equation $M(x,y)dx + N(x,y)dy = 0$ by the substitution $y = vx$. [2]
   (m) Explain why there exists no constant solution of the differential equation $\frac{dy}{dx} = y^2 + 16$. [2]
   (n) Find the solution of $x\frac{dy}{dx} = y^2 - y$ that passes through the point $(2, 1/4)$. [2]

2. (a) Solve the second order linear differential equation $\frac{d^2y}{dx^2} - y = xcos2x + x^2 + e^{3x}$. [4]
   (b) Prove that the transformation $v = y^{1-n} (n \neq 0\ or\ 1)$ reduces the Bernoulli equation $\frac{dy}{dx} = P(x)y + Q(x)y^n$ to a linear equation in v. Hence solve the equation $\frac{dy}{dx} - y = e^x y^2$. [3]
   (c) Find the differential equation corresponding to the family of curves $y = k(x-k)^2$, where k is an arbitrary constant. [2]
   (d) Solve: $\frac{dy}{dx} = tan^2(x+y)$. [1]

3. (a) Solve the differential equation $y = (x-b)p + \frac{a}{p}$. [3]
   (b) Using the method of undetermined coefficients, solve the equation $D^2(D-1)y = 3e^x + sinx$. [4]
   (c) If u and v are any two solutions of the equation $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$ on an interval $[a,b]$, then prove that their Wronskian $W(u,v)$ is either identically zero or never zero on $[a,b]$. [3]

4. (a) Verify that the equation $(2x^2+3x)y'' + (6x+3)y' + 2y = (x+1)e^x$ is exact and then solve it. [4]
   (b) Solve the differential equation $(x-3y+3)dx + (3x+y+9)dy = 0$. [3]
   (c) One solution of the Legendre differential equation $(1-x^2)y'' - 2xy' + 2y = 0$ is $y = x$. Find a second solution. [2]
   (d) Convert the differential equation $y = xf(p) + \phi(p)$ to a linear equation in x. [1]

5. (a) Using the method of variation of parameters, solve the equation $(D^2+1)y = f(x)$, where $f(x)$ is a known integrable function. [3]
   (b) Find the singular solution and extraneous loci of the differential equation $4xp^2 = (3x-5)^2$. [4]

(c) Solve the equation $x^2y'' + xy' - y = x^4e^{3x}$. [3]

6. (a) Solve the differential equation $\frac{dy}{dx} + y = f(x)$ with initial condition $y(0) = 1$, where $f(x) = 1$, when $0 \leq x < 1$ and $f(x) = -1$, when $x \geq 1$. Hence find the value of $y(5)$. [4]

(b) Reduce the equation $(x^2 + y^2)(1 + p)^2 - 2(x + y)(1 + p)(x + py) + (x + py)^2 = 0$ to Clairaut's form by the substitution $x^2 + y^2 = v$ and $x + y = u$. Hence solve the equaion. [3]

(c) Solve: $\frac{d^4y}{dx^4} - cotx\frac{d^3y}{dx^3} = 0$. [2]

(d) Use the method of isoclines to sketch some of the solution curves of the equation $\frac{dy}{dx} = x + y + 1$. [1]

7. (a) If $e^{\int \phi\left(\frac{x^2}{y}\right) d\left(\frac{x^2}{y}\right)}$ is an integrating factor of the differential equation $M(x, y)dx + N(x, y)dy = 0$, then find the expression of $\phi\left(\frac{x^2}{y}\right)$. [3]

(b) Show that if f is any solution of the differential equation $\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$, then the transformation $y = f + \frac{1}{v}$ reduces the above differental equation to a linear in v. Hence solve the equation $\frac{dy}{dx} = -y^2 + xy + 1$; given solution $f(x) = x$. [4]

(c) Find the equation of the family of curves which cut the members of the family of parabolas $y^2 = 4ax$ at angle of 45^o. [3]

B. Sc. (Honours) Examination-2021

Semester-III (CBCS)
Mathematics
Course: SECMA-1
(Boolean Algebra and Circuit Design)

Time: Two Hours **Full Marks: 25**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any five*** questions.

1. (a) Let A be the set of all positive divisors of n and define for $a, b, c \in A$

$$a + b = \text{l.c.m of } a \text{ and } b;$$
$$a.b = \text{g.c.d of } a \text{ and } b;$$
$$\text{and } a' = \frac{n}{a}.$$

[2]

For what values of n, $(B, +, ., ')$ is a Boolean algebra. Find the zero element and the unit element.

(b) Let Y be a non-empty subset of a finite set X. Show that the power sets $P(X)$ and $P(Y)$ are Boolean algebras but $P(Y)$ is not a Boolean subalgebra of $P(X)$. [3]

2. In a Boolean algebra $(B, +, ., ')$, prove that for all a, b, $c \in B$,

(a) $a + b = a + c$ and $a.b = a.c \Rightarrow b = c$, [2]

(b) $a.b.c + a.b.c' + a.b'.c + a'.b.c = a.b + b.c + c.a.$ [3]

3. Define Boolean function of n variables. Find the function f of three variables x, y, z such that

$$f(x, y, z) = 0 \text{ if two of the variables are 1 and the other is 0,}$$
$$= 1 \text{ otherwise.}$$

Express the expression in disjunctive normal form. [1+4]

4. Minimize the function $f(x, y, z, t) = x'yz + x'z't' + xy't' + xyt + yz't$ using Karnaugh map. [5]

5. Convert the following numbers as directed.
(a) $(1729)_{10} = (\quad)_8$, (b) $(100110)_2 = (\quad)_{10}$, (c) $(203)_{16} = (\quad)_3$. [$1\frac{1}{2}$+$1\frac{1}{2}$+2]

6. Add and subtract the numbers $(1010010)_2$ and $(11011)_2$. [2+3]

7. Draw the circuit which realises the Boolean expression $xyz + xy'z + xy'z'$ and simplify the circuit, if possible. [5]

8. (a) Find the output sequence Y for an AND gate with inputs A, B, C
$A = 10111001$; $B = 10101010$; $C = 01101100$. [1]

(b) Find the output sequence Y for an OR gate with inputs A, B, C
$A = 11001010$; $B = 01100110$; $C = 00101000$. [1]

(c) Draw the circuit which realises the Boolean expression $AB' + (A + BC)' + (A'B + C')$ using 'OR-AND' gates. [3]