Use separate answer script for each unit

B. Sc. (Honours) Examination-2021

Semester-V (CBCS)
Mathematics
Course: CCMA-11
(Analysis-V and Differential Equations-III)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Analysis-V)
(Full Marks: 30)

Answer ***any three*** questions.

1. (a) A function f is bounded and integrable on $[a, b]$ for every $b > a$ such that the improper integral $\int_a^b f dx$ is convergent. Does $\lim_{x\to\infty} f(x)$ exist? Justify. Find the value of $\lim_{x\to\infty} f(x)$ if/whenever it exists. [2+5]

   (b) If f and g are two positive functions on $[a, b]$ such that $\lim_{x\to a+} \frac{f(x)}{g(x)} = 0$, then show that $\int_a^b f dx$ converges, provided $\int_a^b g dx$ converges. [3]

2. (a) Let f_n be uniformly continuous on an interval $I \subseteq \mathbb{R}$, for $n = 1, 2, \ldots$ such that $f_n \longrightarrow f$ uniformly on I. Is f uniformly continuous on I? Justify. [5]

   (b) For each $n \geq 1$, suppose that the function f_n is continuous on an interval $I \subseteq \mathbb{R}$ and $f_n \longrightarrow f$ uniformly on I. Let $\{x_n\}$ be a sequence in I which converges to $x \in I$. Discuss the convergnce of the sequence $\{f_n(x_n)\}$. [5]

3. (a) Given a convergent sequence $\{a_n\}$, determine whether the series $\sum_{n=1}^{\infty} a_n \cos nx$ is uniformly convergent on $\mathbb{R}$. [4]

   (b) Prove that the series $\sum(-1)^n x^n(1-x)$ converges uniformly on $[0, 1]$. Discuss the uniform convergence of $\sum x^n(1 - x)$. [3+3]

4. (a) Define the radius of convergence and the interval of convergence of a power series $\sum a_n x^n$. Determine the interval of convergence for the power series $\sum a_n x^n = (x - 1) + \frac{(2!)^2}{4!}(x - 1)^2 + \frac{(3!)^2}{6!}(x - 1)^3 + \cdots$. [2+4]

   (b) Let f be the sum of a power series $\sum a_n x^n$. Prove or disprove: The function f is differentiable on the interval of convergence of $\sum a_n x^n$. [4]

5. (a) For which functions, the Fourier series are defined in general? Define the Fourier series and Fourier coefficients for such a function f. Does the Fourier series of f converge to the function f? [4]

   (b) Find the Fourier series of the periodic function $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ (with period 2π) defined by

$$f(x) = \begin{cases} 1 & \text{for } -\pi < x \leq 0 \\ 2x + 1 & \text{for } 0 \leq x \leq \pi. \end{cases}$$

   Also find the sum of the series at $x = 0, \pm\pi, \pm 2\pi$. [6]

Unit-II (Differential Equations-III)
(Full Marks: 30)

Answer question number 1 and ***any two*** from the rest.

1. Answer ***any five*** questions.
   (a) What is the difference between a linear and nonlinear partial differential equation (PDE)?
   (b) Find a partial differential equation by eliminating arbitrary function ϕ from the equation $x^2 + y^2 = \phi(yz - y^2)$; $z = z(x, y)$.
   (c) Define the singular integral of the first-order PDE in two independent variables.
   (d) Solve the PDE $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2xy$, together with the auxiliary condition that $z(x, x^2) = 2$.
   (e) Classify compatible systems of first-order PDEs in two independent variables.
   (f) Show that the complete integral of the PDE $z = (x - 7)\frac{\partial z}{\partial x} + (y - 9)\frac{\partial z}{\partial y}$ represents all possible planes passing through the point $(7, 9, 0)$.
   (g) Find the particular integral (PI) of the PDE
   $$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = log_e(2y - 3x).$$
   (h) Find two initial strips of the following PDE, which passes through the x-axis in the context of Cauchy's method of characteristic:
   $$2z = [(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2] + (2\frac{\partial z}{\partial x} - x)(2\frac{\partial z}{\partial y} - y).$$
   [5×2 = 10]
2. (a) If $z(x, y)$ is the solution of the Cauchy problem $x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2021$, $z(x, 0) = -x^2$, $x > 0$, then find the value of $z(1, 2021)$. Is the solution unique? Justify your answer.
   (b) Show that the PDEs $x\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + y(\frac{\partial z}{\partial y})^2 - 1 = 0$ and $\frac{\partial z}{\partial x} - 9\frac{\partial z}{\partial y} = 0$ are compatible on a specified domain and hence find the one-parameter family of common solutions. [(3+1+1)+(2+3)]
3. (a) For the PDE
   $$(z^2 - 2yz - y^2)\frac{\partial z}{\partial x} + x(y + z)\frac{\partial z}{\partial y} = x(y - z),$$
   find the general integral and the integral surface through the curve $x = 2y = 3z$.
   (b) Find the singular integral of the following PDE:
   $$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + 3\sqrt[3]{\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}}.$$
   [(3+2)+5]
4. (a) Using Cauchy's method of characteristics, find an integral surface of the PDE $x(\frac{\partial z}{\partial x})^2 + y\frac{\partial z}{\partial y} = z$ passing through the straight line $x_0 = s$, $y_0 = 1$, $z_0 = -s$, $0 < s < 1$.
   (b) Solve:
   $$2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial y} = 3cos(3x - 2y).$$
   [5+5]

B.Sc. (Honours) Examination, 2021

Semester-V (CBCS)
Mathematics
Course: CCMA-12
(Numerical Analysis)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any six*** questions.

1. (a) Explain (with suitable examples) what do you mean by inherent errors and round off errors. [4]

   (b) If $u = xyz^2 + \frac{3}{2}x^3y^5$ and errors in x, y, z are $0.005, 0.001, 0.001$ respectively at $x = 3, y = z = 1$, compute the maximum absolute and relative errors in evaluating u. [4]

   (c) Find the sum of the approximate numbers $2.56, 4.5627, 1.253, 1.0534$ if the numbers are correct to the last digit. Give proper justification in support of your answer. [2]

2. (a) Given $y_0 = 2, y_1 = 11, y_2 = 80, y_3 = 200, y_4 = 100, y_5 = 8$, find $\nabla^5 y_5$ without using the difference table. [3]

   (b) Show that if Δ operates on n, then $\Delta\left[\binom{n}{x+1}\right] = \binom{n}{x}$ and hence $\sum_{n=1}^{N}\binom{n}{x} = \binom{N+1}{x+1} - \binom{1}{x+1}$. [4]

   (c) Prove that $\Delta + \nabla \equiv \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$. [3]

3. (a) What do you understand by interpolation? Establish Newton's interpolation formula using forward differences when the functional values $y = f(x)$ are known at $(n+1)$ equispaced points. [5]

   (b) Compute the value of x for $y = 6.303$ by using the following table:

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7
y	2.631	3.328	4.097	4.944	5.875	6.836	8.013

[5]

4. (a) State principle of numerical differentiation. Deduce numerical differentiation formula from Newton's backward interpolation formula at an interpolating point. Also derive the error involved in this formula. [1+3+3]

   (b) Show that $y' = \frac{1}{h}\left[\Delta y - \frac{1}{2}\Delta^2 y + \frac{1}{3}\Delta^3 y - \frac{1}{4}\Delta^4 y + \cdots\right]$. [3]

5. (a) Establish Simpson's $\frac{1}{3}$rd rule from Newton-Cotes quadrature formula (closed type). Estimate the error involved in this rule. Explain geometrically why does this rule call a parabolic rule. [2+4+2]

   (b) How big should the spacing (h) be so that the error in computing $\int_2^7 \frac{dx}{x}$, using the Trapezoidal rule, is less than or equal to 0.5×10^{-5}? [2]

6. (a) Show that in secant method for root finding, the error in the $(n+1)$th iterate is approximately proportional to the product of the errors of the nth and the $(n-1)$th iterate. Hence show that the order of convergence for secant method is $\frac{1+\sqrt{5}}{2}$. [4+3]

   (b) Discuss the convergence/divergence of fixed point iteration method geometrically in solving one real root of the equation $f(x) = 0$ or $x = \phi(x)$, for the cases $0 < \phi'(x) < 1$, $-1 < \phi'(x) < 0$, $\phi'(x) > 1$ and $\phi'(x) < -1$. [3]

7. (a) Show how to use Newton-Raphson method to solve the equation below for x:

$$\int_{-x}^{x} e^{\sin t}\,dt = 1$$

   Use the Composite Simpson's rule with $n = 4$ intervals for the integral. You do not need to perform any numerical calculations. (**Hint:** *In order to find out the derivative term in Newton-Raphson method, use* $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$.) [6]

(b) Determine α, β and γ such that the formula

$$\int_0^h f(x)dx = h\left[\alpha f(0) + \beta f\left(\frac{h}{3}\right) + \gamma f(h)\right]$$

is exact for polynomials of as high order as possible. [4]

8. (a) Let $P_3(x)$ be the interpolating polynomial for the data $(0,0), (0.5, y), (1,3)$ and $(2,2)$. The coefficient of x^3 in $P_3(x)$ is 6. Find y. [3]

(b) Solve the equation $\frac{dy}{dx} = x - y^2$ with the condition $y(0) = 1$ by Taylor's series method to find $y(0.1)$ correct to 3 decimal places. [3]

(c) The function defined by $f(x) = \sin \pi x$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the Bisection method converges to (i) 0, if $a + b < 2$ (ii) 2, if $a + b > 2$ (iii) 1, if $a + b = 2$. [4]

9. (a) What do you mean by complete pivoting? Carry out the total operational count required for solving a system of n linear equations in n unknowns by Gauss-Jordan elimination method. [5]

(b) Solve the following system of equation by Gauss-Seidel iteration method correct to 3 decimal places.

$$\begin{bmatrix} 3 & 9 & -2 \\ 4 & 2 & 13 \\ 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 24 \\ -8 \end{bmatrix}$$

[5]

10. (a) Let y_n be an approximate solution of the first order initial value problem $y' = f(x,y), y(x_0) = y_0$, generated by the first order Euler method. If the exact solution of the given differential equation has a continuous second derivative on the interval $[x_0, b]$, and if on this interval the inequalities $|f_y(x,y)| \le L$ and $|y''(x)| < Y$ are satisfied for fixed positive constants L and Y, the error ε_n of Euler's method at a point $x_n = x_0 + nh$ is bounded as follows:

$$|\varepsilon_n| \le \frac{hY}{2L}(e^{(x_n - x_0)L} - 1).$$

Here is the question related to the above result: Estimate the step size h for Euler's method needed to make the absolute value of the error less than $0.25e^{-1}$, given the bounds $|f(x,y)| < 1.387$, $|f_x(x,y)| < 0.6795|x|$, $|f_y(x,y)| < 0.5432|x|$ on $[-2.25, 1.35]$. [5]

(b) Compute $y(0.3)$, from the equation $\frac{dy}{dx} = \frac{0.5 - x + y^2}{1 + y + x^2}, y(0) = 0$, taking $h = 0.1$, by Runge-Kutta method of order four (RK-4), correct to five decimal places. [5]

Use separate answer script for each unit

B.Sc. (Honours) Examination-2021

Semester-V (CBCS)
Mathematics
Course: DSEMA-1
(Mechanics-II)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 40)
(Dynamics of a rigid body)
Answer ***any four*** questions.

1. (a) Define centre of percussion for rigid body motion. [1]

(b) A pendulum is constructed of a solid sphere of mass M and radius a. This is attached to the end of a rod of mass m and length b. Show that there will be no stress on the axis, if the pendulum be struck at a distance

$$\frac{M\{\frac{2}{5}a^2+(a+b)^2\}+\frac{1}{3}mb^2}{M(a+b)+\frac{1}{2}mb}$$

from the axis. [5]

(c) A uniform beam AB of length 2a can turn about its end A and is in equilibrium. Show that the points of its length where a blow must be applied to it, so that the impulses at A may be in each case $\frac{1}{n}$-th of that blow are $\frac{4}{3}\frac{n\pm1}{n}a$. [4]

2. (a) If the moments and products of inertia of a rigid body about three mutually perpendicular axes OXYZ be known, then find the moment of inertia about any other axis OQ through their meeting point. Hence find the momental ellipsoid of a body about the origin O. [5]

(b) Show that the equation of the momental ellipsoid at the centre of an elliptic plate is

$$b^2x^2+a^2y^2+(a^2+b^2)z^2=constant.$$

[5]

3. (a) Find the kinetic energy of a rigid body moving in two dimensions in the form

$$\frac{1}{2}Mv^2+\frac{1}{2}Mk^2\dot{\theta}^2$$

where M is the mass, v is velocity of the centre of mass, θ is the angular velocity about the centre of mass. [5]

(b) A uniform hollow cylinder of small thickness is placed with its axis horizontal on a plane whose inclination to the horizon is α. Find the least coefficient of friction between it and the plane for pure rolling. [5]

4. (a) For a rigid plane lamina lying in the XY-plane, if $\angle XOX'=\theta$, show that the product of inertia about the lines OX' and OY' perpendicular to OX' in the plane, is

$$(A-B)sin\theta cos\theta+Hcos2\theta$$

Deduce that OX' and OY' are two principal axes at O if θ satisfies the equation

$$tan2\theta=\frac{2H}{B-A}$$

Also determine the inclinations of the principal axes at the vertex O to the side OA of the rectangle OABC of sides 2a and 2b. [6]

(b) Show that the principle axes at the node of a half-loop of the lemniscate $r^2=A^2cos2\theta$ are inclined to the initial line at angles

$$\frac{1}{2}tan^{-1}\frac{1}{2}\ \text{ and }\ \frac{\pi}{2}+\frac{1}{2}tan^{-1}\frac{1}{2}$$

[4]

5. (a) Find the minimum time of oscillation of a compound pendulum. [3]

   (b) Show that centre of suspension and oscillation of a compound pendulum are interchangeable. [3]

   (c) A solid homogeneous cone, of height h and vertical angle 2α, oscillates about a horizontal axis through its vertex. Find the length of the simple equivalent pendulum. [4]

6. (a) Prove that the rate of change of the angular momentum of the body about the axis of rotation is equal to the sum of the moments about the same axis of all forces acting on the body. [5]

   (b) A fine string has two masses M and M' tied to its ends and passes over a rough pulley, of mass m and radius a, whose centre is fixed. If the string does not slip over the pulley, then show that M will descend with an acceleration equal to

$$\frac{(M - M')a^2 g}{(M + M')a^2 + mk^2},$$

   where k is the radius of gyration of the pulley. [5]

Unit-II (Full Marks: 20)

(Hydrostatics)

Answer ***any two*** questions.

1. (a) Define shear stress and normal stress for a fluid in equilibrium. [1+1]

   (b) Show that the surface of separation of two liquids of densities ρ and σ $(\rho > \sigma)$, at rest under gravity is a horizontal plane. [4]

   (c) A fine parabolic tube is held with its axis vertical and vertex downwards and is filled with three liquids whose densities are σ_1, σ_2 and σ_3 beginning from the top of one side. If R_1, R_2, R_3, R_4 be the distances of the boundaries from the focus, prove that $\sigma_1 R_1 + (\sigma_2 - \sigma_1) R_2 + (\sigma_3 - \sigma_2) R_3 = \sigma_3 R_4$. [4]

2. (a) If $\vec{F}$ denotes the external force per unit mass at any point (x, y, z) in a fluid, show a necessary and sufficient condition of equilibrium is $\vec{F} \cdot (\vec{\nabla} \times \vec{F}) = 0$. [6]

   (b) A given volume V of liquid is acted upon by forces $-\frac{\lambda x}{a^2}, -\frac{\lambda y}{b^2}, -\frac{\lambda z}{c^2}$ per unit mass at (x, y, z) parallel to the axes. Find the equation of the free surface. [4]

3. (a) Define centre of pressure of a given area in a fluid in equilibrium. Show that the depth of centre of pressure (C.P) of a plane area immersed in a homogeneous liquid is greater than the depth of its centroid or centre of gravity (C.G). [1+4]

   (b) A hollow weightless cone of semi-vertical angle γ and of height H is filled with liquid of density ρ and freely hung from a point on the rim of the base. Find the ratio of the thrust on the base to the weight of the liquid contained in the cone. [5]

Use separate answer script for each unit

B.Sc. (Honours) Examination-2021

Semester-V (CBCS)
Mathematics
Course: DSE-2

(Linear Programming Problem, Game Theory and Mathematical Statistics)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 30)
(Linear Programming Problem (LPP), Game Theory)

Answer ***any three*** questions.

1. (a) Define convex set. Test whether the set $X = \{(x, y) \subset \mathbb{R}^2 : \ y \leq |x|\}$ is convex or not. [1+2]

   (b) Verify whether optimum solution of the LPP
   Maximize $Z = x_1 + x_2$
   subject to

$$\begin{aligned} 2x_1 - x_2 &\geq -1 \\ x_1 &\leq 2 \\ x_1 + x_2 &\leq 3 \\ x_1, x_2 \geq 0 \end{aligned}$$

   exists or not. Is that unique, if exists? [4]

   (c) Consider the system of equations

$$\begin{aligned} 2x_1 + x_2 - 5x_3 = 20 \\ 4x_1 - 3x_2 - 10x_3 = 15. \end{aligned}$$

   Find, if possible, the basic solution with x_1 as non-basic variable. [3]

2. (a) Check whether $(2, 3, 1)$ is a basic feasible solution of the system of equations

$$\begin{aligned} 2x_1 + x_2 + 4x_3 = 11 \\ 3x_1 + x_2 + 5x_3 = 14. \end{aligned}$$

   Find other basic feasible solutions, if exists. [5]

   (b) Solve the LPP
   Minimize $Z = -4x_1 - 10x_2$
   subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 50 \\ 2x_1 + 5x_2 &\leq 100 \\ 2x_1 + 3x_2 &\leq 90 \\ x_1, x_2 \geq 0 \end{aligned}$$

   by the simplex method, if possible. [5]

3. (a) Obtain the dual of the LPP
Maximize $Z = x_1 + x_2 + x_3$
subject to

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 5 \\ x_1 - 2x_2 &\geq 3 \\ 2x_2 - x_3 &\leq 4 \end{aligned}$$

$x_1, x_2 \geq 0$ and x_3 is unrestricted in sign.

[4]

(b) Solve the following LPP by two-phase method.
Maximize $Z = 5x_1 + x_2 - 2x_3 + x_4$
subject to

$$\begin{aligned} x_1 + 5x_2 - 8x_3 + 3x_4 &= 6 \\ 3x_1 - x_2 + x_3 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

[6]

4. (a) Find the optimal solution and the minimum cost of the transportation problem

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	4	30
O_2	3	4	7	8	15
O_3	9	6	5	8	15
b_j	10	25	18	7	

.

[5]

(b) Obtain the optimal assignment and minimum cost for the assignment problem

	I	II	III	IV	V
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

.

[5]

5. (a) Define pure strategy and mixed strategy of a game. [2]

(b) Determine value of a so that the game with the following pay off matrix is strictly determinable.

λ	7	3
-2	λ	-8
-3	4	λ

[3]

(c) Solve the game problem graphically

	B_1	B_2
A_1	2	-1
A_2	3	2
A_3	-1	5
A_4	-2	1

.

[5]

Unit-II (Full Marks: 30)
(Mathematical Statistics)
Answer ***any three*** questions.

1. (a) Define sampling distribution of a statistic. Find the sampling distribution of the mean for the gamma population. [1+2]
   (b) Find the variance of the sample variance S^2 for a random sample drawn from a normal (m, σ) population. Show that S^2 is a consistent but biased estimate of σ^2. [3+2+2]

2. (a) Find the distribution of the smallest sample value in a sample of size n from the population whose distribution is given by

$$dF = e^{-x}dx,\ 0 \leq x \leq \infty.$$

   Also, find the mean and variance of the distribution. [4+1+1]
   (b) Prove that the maximum likelihood estimate of the parameter λ of a population having density function $2(\lambda - x)/\lambda^2$, $0 < x < \lambda$, for a sample of unit size is $2x$, x being the sample value. Hence show that the estimate is biased. [3+1]

3. (a) Let $x_1,\ x_2, ..., x_n$ be a random sample of size n drawn from a $N(0, \sigma)$ population. Show that the statistic $\sum_{i=1}^{n} \frac{x_i^2}{n}$ is an unbiased estimate of σ^2. Examine whether this estimate is consistent. [2+4]
   (b) Using the Neyman-Pearson theorem construct a test of the null hypothesis $H_0 : m = m_0$ against an alternative $H_1 : m = m_1$ for a $N(m, \sigma)$ population where σ is known and $m_0 > m_1$. [4]

4. (a) In the framework of hypothesis test, define Type I and Type II errors. Hence define the level of significance and power of the test. [(2+2)+(1+1)]
   (b) Let p denote the probability of getting a head when a given coin is tossed once. Suppose that the hypothesis $H_0 : 5 = 0.5$ is rejected in favor of $H_1 : p = 0.6$ if 10 trials result in 7 or more heads. Calculate the probabilities of Type I and Type II errors. [2+2]

5. (a) The weights in gram of a sample of 12 items are 7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 taken at random from its population which is normal having standard deviation 5. Find 95% confidence limits for the mean of the population. Given that

$$\frac{1}{\sqrt{2\pi}} \int_{1.96}^{\infty} e^{-x^2/2} dx = 0.025.$$

   [4]
   (b) Find by the method of likelihood ratio testing a test for the null hypothesis $H_0 : m = m_0$ for a $N(m, \sigma)$ population when σ is known. [6]