M. Sc. Examination-2022

Semester-II

Mathematics

Course: MMC-21

(Functional Analysis)

Time: Three Hours | Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any <u>four</u> questions

1. (a) Prove that every compact subset in a metric space is closed and bounded. Does the converse hold? Support your answer. [4]

   (b) Define Lebesgue number. Prove that in a sequentially compact metric space every open cover has a Lebesgue number. [4]

   (c) Show that intersection of finite number of compact subsets in a metric space is compact. [2]

2. (a) State Riesz Lemma and hence show that if a closed unit ball in a normed linear space X is compact then X is finite dimensional. [4]

   (b) Show that $C[0,1]$ is a Banach space with respect to a norm to be defined by you. [4]

   (c) A function $||\ ||$ is defined on $\mathbb{R}^2$ by $||x|| = \max\{|x_1|, |x_2|\}$ where $x = (x_1, x_2)$. Verify whether $||\ ||$ is a norm or not on $\mathbb{R}^2$. [2]

3. (a) When is a linear operator over a normed linear space said to be bounded? Prove that every linear operator defined on a normed linear space is continuous iff it is bounded. [4]

   (b) Show that norm($||\ ||$) of a bounded linear operator T is given by

$$||T|| = \underset{x\in X\setminus\{\theta\}}{Sup} \frac{||Tx||}{||x||}.$$

[3]

   (c) Consider the normed linear space $C[0,1]$ with respect to Sup norm. Let $t_0 \in [0,1]$be a fixed element and define a functional f on $C[0,1]$ by $f(x) = x(t_0)$, for all $x \in C[0,1]$. Prove that f is a bounded linear functional and find $||f||$. [3]

4. (a) State Hahn-Banach theorem in a normed linear space and use it to show that if x_0 is a non-zero vector in X, then there is a member $f \in X^*$ such that $f(x_0) = ||x_0||$ with $||f|| = 1$. Hence verify that X^* distinguishes points of X. [1+4+1]

(b) Show that every finite dimensional normed linear space is complete. [4]

5. (a) In an inner product space $(X, < \; >)$, prove that

$$| < x, y > | \leq ||x|| \, ||y||, \; \forall x, y \in X$$

where $||x||^2 =< x, x >$ and hence show that X is a normed linear space with respect to the norm induced by the inner product. [5]

(b) Prove that inner product is a continuous function. [2]

(c) Show that $\mathbb{R}^n$ is an inner product space with respect to an inner product to be defined you. [3]

6. (a) Prove that a Banach space X is a Hilbert space iff parallelogram law holds in X. [5]

(b) Define orthonormal set in an inner product space. Show that any orthonormal set of vectors in an inner product space is linearly independent. [3]

(c) Prove that in an inner product space X,

$$x \perp y \text{ iff } ||x + \alpha y|| = ||x - \alpha y||$$

for all $x, y \in X$ and for any scalar α. [2]

M. Sc. Examination-2022

Semester-II

Mathematics

Course: MMC-22

(Topology)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Write down Zorn's lemma and Hausdorff maximal principle. Show that Zorn's lemma implies Hausdorff maximal principle. [2+3]
   (b) Show that a compact subset of a Hausdorff space is closed. [3]
   (c) Show that a metric space is normal. [2]

2. (a) Define a sub-basis for a topology on a non-empty set X. Describe how you can derive a topology from a given sub-basis on X. [1+3]
   (b) What are rays in an ordered set X? Show that the open rays form a sub-basis for the order topology on X. [1+3]
   (c) Let τ and τ' be two topologies on a non-empty set X. Explain 'τ' is finer than τ' in terms of the basic open sets. [2]

3. (a) Construct a subspace X of the real line such that $[3, 5)$ and $[8, 9]$ are open sets whereas $(6, 7)$ is closed in X. Justify your answer. [3]
   (b) Let Y be a subset of a topological space X. What do you mean by its closure $\overline{Y}$ in X. Show that $y \in \overline{Y}$ if each basic open set containing y intersects Y. [1+3]
   (c) Show that a set A is closed in a subspace X of a topological space Y if and only if $A = B \cap X$, where B is a closed subset of Y. [3]

4. (a) Give definition of T_2 space and T_3 space. Show that a T_3 space is a T_2 space but the converse is not true. [2+2+3]
   (b) Show that a subspace of a completely regular space is completely regular. [3]

5. (a) Show that every continuous image of a compact set is compact. [2]
   (b) Show that $\mathbb{R}$ with lower limit topology is not 2nd countable. [2]
   (c) Let X be a 1st countable space, $A \subset X$ and $a \in X$. then show that $a \in \overline{A}$ if and only if there is a sequence $\{x_n\}$ in A such that $\lim_{n \to \infty} x_n = a$.
   Also show that "only if" part of the above is not true without the space X being 1st countable. [4+2]

6. (a) Let $\{X_i\}$ be a non-empty family of connected subsets of a topological space Y such that $X_i \cap X_j \neq \phi$ for all i, j with $i \neq j$. Prove or disprove: $\bigcup_i X_i$ is a connected subset of Y. [3]
   (b) Let X be a topological space, $Y = \{x \in \mathbb{R} : x^2 = 5\}$ and $f : X \longrightarrow Y$ be a non-constant continuous function. Show that X cannot be connected. [3]
   (c) What are components of a topological space X? Determine whether $X = Y \cup Z$ is connected, where $Y = \{(x, \sin\frac{1}{x}) : 0 < x \leq 1\}$ and $Z = \{(0, y) : -1 \leq y \leq 1\}$. Find the components of X. [4]

M. Sc. Examination-2022

Semester-II
Mathematics
MMC-23
(Abstract Algebra)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Show that $G = \mathbb{Z}_{10} \times \mathbb{Z}_{21}$ contains an element of order 210. Also find total number of generators of G. How many elements of G are of order 7? [3+1+1]

   (b) Show that the group $\mathbb{Q}$ can not be expressed as an internal direct product of two nontrivial subgroups. [2]

   (c) Let G be a finite p-group and S be a finite G-set. If $S_0 = \{a \in S \mid ga = a \text{ for all } g \in G\}$, then show that $|S| \equiv |S_0| (mod\ p)$ [3]

2. (a) Let G be a group and $a \in G$. Prove that $|C_l(a)| = [G : C(a)]$. State and prove the conjugacy class equation. [2 + 2]

   (b) State and prove the converse of the Lagrange's theorem for finite abelian groups. [3]

   (c) Let G be a group such that $[G : Z(G)] = n$. Show that no elements of G can have more than n conjugates. [3]

3. (a) State and prove the Sylow's first theorem. [4]

   (b) Show that every group of order 33 is cyclic. [3]

   (c) Prove that no group of order 56 is simple. [3]

4. (a) Let R be a ring with unity. Then show that $\frac{R[x]}{<x>} \simeq R$. [3]

   (b) Find all units of $\mathbb{Z}_8[x]$. [2]

   (c) Show that $\mathbb{Z}[x]$ is not a PID. Give an example of a PID which is not ED. [2+1]

   (d) Let $(E, +, \cdot, v)$ be an ED. Then for $a \in E$, show that $v(a) = v(-a)$. [2]

5. (a) Show that $\mathbb{Z}[\sqrt{2}]$ has no unit between 1 and $1 + \sqrt{2}$. [4]

   (b) In the ring $\mathbb{Z}[i\sqrt{5}]$ find GCD of 2 and $1 + i\sqrt{5}$. [3]

   (c) In $\mathbb{Z}[i]$, if $a + bi$ is an element such that $a^2 + b^2$ is a prime integer, then show that $a + bi$ is a prime element. [3]

6. (a) In a UFD show that every irreducible element is prime. [4]

   (b) Find all irreducible polynomials of degree 2 over $\mathbb{Z}_2$ [3]

   (c) Check the irreducibility of the polynomial $x^6 + x^3 + 1$ over $\mathbb{Z}$. [3]

M. Sc. Examination-2022

Semester-II

Mathematics

Course: MMC-24

(Classical Mechanics)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. Discuss the principle for deriving forces of constraints exerted by holonomic and nonholonomic constraints. Derive Lagrange's equation of motion of the first kind for a system of $N(> k + l)$ particles having k-holonomic and l-nonholonomic constraints. Find constraints and the forces of constraints on a point mass suspended from a ceiling, oscillating around the stable equilibrium state on a vertical plane passing through the point of suspension. [3+3+4]

2. State and prove a theorem on Legendre transformation in its general form. Derive Hamilton's equation of motion from it. Find the Hamiltonian for free motion of a particle on a spherical surface of radius a. [3+4+3]

3. Define Hamilton's principal function and derive an equation for it. Examine whether the derived equation is separable for the harmonic oscillator problem. If yes, find the ordinary differential equation for the x-dependent part of the function. [1+4+2+3]

4. Explain normal coordinates and frequencies of small oscillations of a mechanical system changing state around one of its stable equilibrium states. Find the equilibrium point, normal coordinates, and normal frequency of the system whose Hamiltonian is $H = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2 + \frac{k_1}{2}(q^1 - a)^2 + \frac{k_2}{2}(q^2 - b)^2 + \frac{k_3}{2}(q^3 - c)^2\right)$. [6+4]

5. Obtain the equation of motion of a particle suspended from the roof, considering the earth's rotation about its axis. Hence show that the oscillation around the close vicinity of the stable equilibrjum of the particle rotates in a horizontal plane. Explain whether the rotation's sense (clockwise or anticlockwise) depends on the region(southern or northern hemisphere) of the point of suspension on the earth. [5+4+1]

6. A symmetric rigid body having a point fixed on its axis of symmetry is rotating freely. Find the equation for the change in the angular velocity $\vec{\omega}(t)$. Verify whether or not $|\vec{\omega}(t)|$ changes with time. Also, check whether $\vec{\omega}(t)$, $\vec{\Omega}(t)$ and the axis of symmetry of the body lying on a plane throughout the motion. Hence discuss how the space- and the body cones involve in the movement. [3+1+2+4]

Use separate answer script for each unit

M. Sc. Examination-2022

Semester-II
Mathematics
Paper : MMC-25 (New & Old Syllabus)
(Solid Mechanics and Dynamical Systems)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 20)

(Solid Mechanics)
Answer *any two* questions.

1. (a) Prove that $E_{ij}(i \neq j); i, j = 1, 2, 3$ denote decrease in right angle between two orthogonal line elements (E'_{ij}s are the Lagrangian infinitesimal strain components). [5]

   (b) The displacement in an elastic solid is given as follows:

$$u_1 = \epsilon(X_1 + 2X_2 + 3X_3)$$
$$u_2 = \epsilon(-2X_1 + X_2)$$
$$u_3 = \epsilon(X_1 + 4X_2 + 2X_3),$$

   where ϵ is very small. Calculate dilatation, rotation and find principal strains. [5]

2. (a) Define stress quadric. Prove that the normal stress across any plane through the centre of stress quadric is proportional to the inverse of the square of the central radius vector of the quadric normal to the plane. [1+4]

   (b) The stress tensor at a point is given by
   $T_{ij} =$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & b & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

   where b is a constant. Determine b so that the stress vector on some plane at the point will be zero. Determine the direction cosines of the normal to that plane. [5]

3. State the principle of conservation of energy for a thermo-mechanical continuum and hence obtain the Helmholtz's equation of heat conduction. [2+8]

Unit-II (Full Marks: 20)

(Dynamical Systems)
Answer *any two* questions.

1. (a) Show that the cycles $\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$ and $\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\}$ form two unstable periodic-3 cycles of the tent map. [2]

   (b) Show that the tent map and the logistic map are topologically conjugate with respect to the map $h : [0, 1] \to [0, 1]$ defined by $h(x) = (1 - \cos \pi x)/2 = \sin^2(\pi x/2)$. [3]

   (c) Show that $\{-1, 1\}$ is an attracting 2-cycle of the map $f(x) = -x^{1/3}, x \in \mathbb{R}$. Also find the basin of attraction and basin boundary of the fixed point of the map. [2]

   (d) Define the Euler shift map $S(x)$. Obtain the fixed points of $S^2(x)$. [3]

2. (a) Find the periodic-2 points of the 2D Hénon map $(x, y) \to H_{a,b}(x, y) = (a - x^2 + by, x)$. Show that a periodic-2 point of the Hénon map exists iff $4a > 3(1 - b)^2$. [2+2]

   (b) Find the potential of the 2D system $\dot{x} = 2xy + y^3, \dot{y} = x^2 + 3xy^2 - 2y$. [2]

   (c) By using Bendixson's Negative criteria, show that the equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ cannot have periodic solution whose phase path lies in a region, where f is of same sign. [4]

3. (a) State Liénard Theorem. Hence show that the equation $\ddot{x} + \mu(x^6 - 1)\dot{x} + x = 0, \mu > 0$ has a unique stable limit cycle, provided $\mu > 0$. [1+3]

   (b) When a fixed point is said to be super stable? Give an example. [1+1]

   (c) If $f(x) = x^2, x \in \mathbb{R}$ is an 1D map, then find it's eventually fixed points (if any). [2]

   (d) Find all periodic two orbits of the map $f(x) = 4x(1 - x), x \in [0, 1]$. [2]

M.Sc. Examination, 2022

Semester-II

Mathematics

Core Course: MMC-26 (New & Old Syllabus)

Numerical Analysis

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) If $f(x) = (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)$, then prove that $f[x_0, x_1, x_2, \cdots, x_n, x, z] = 0$ for all x and z. [4]

   (b) Define confluent divided difference. Show that $f[x, x, \cdots, x, x_0, x_1, \cdots, x_n] = \frac{1}{k!}\frac{d^k}{dx^k} f[x, x_0, x_1, \cdots, x_n]$ where x occurs $(k+1)$ times in the argument of $f[x, x, \cdots, x, x_0, x_1, \cdots, x_n]$. [2+4]

2. (a) How Hermite polynomial of degree $2n+1$ is generated using Newton's divided difference interpolation formula on the $(n+1)$ distinct numbers $x_0, x_1, \cdots, x_n$ for the function f? Explain in details. [4]

   (b) If S is a natural cubic spline that interpolates a twice continuously differentiable function f at the knots $a = x_0 < x_1 < \cdots < x_n = b$, then show that $\int_a^b [S''(x)]^2 dx \leq \int_a^b [f''(x)]^2 dx$. What is the physical significance of this result? [5+1]

3. (a) If $P_n(x)$ is any monic polynomial of degree n and $\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x)$ is the monic Chebyshev polynomial, then show that $\max_{x \in [-1,1]} |\tilde{T}_n(x)| < \max_{x \in [-1,1]} |P_n(x)|$. [5]

   (b) Find the sixth Maclaurin polynomial for e^x, and use Lanczos economization method to obtain a lesser-degree polynomial approximation with a tolerance $\epsilon = 0.05$ on $[-1, 1]$. [5]

4. (a) What do you mean by closed-type numerical quadrature formula? If $f \in C^{n+2}[a, b]$ then show that for $a < \xi < b$, the error committed in the closed-type Newton-Cotes quadrature formula, for n even is

$$E_{NC} \cong \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_0^n u^2(u-1) \cdots (u-n) du.$$

[1+6]

   (b) Evaluate the integral $I = \int_0^1 \frac{dx}{1+x}$ by subdividing the interval $[0, 1]$ into two equal parts and then using the Gauss-Legendre three-point formula $\int_{-1}^1 f(x)dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$. [3]

5. (a) Explain briefly the power method for the largest eigen pair of a real numerical matrix including its stopping criterion. State two major disadvantages of the power method. [4+2]

   (b) Transform the matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

   to tridiagonal form by Householder's method and hence find all the eigenvalues. [4]

6. (a) What do you mean by propagation problem? Explain with a suitable example. [2]

   (b) Find a backward difference approximation of $O(\Delta x)$ for $\frac{\partial^4 u}{\partial x^4}$. [3]

   (c) Write down the FTCS scheme for approximating the parabolic equation $u_t = a u_{xx}$ at the $(i, n)^{th}$ point with a being positive constant. Hence investigate the consistency of the FTCS scheme. [5]