M.Sc. Examination - 2022
Semester-III
Mathematics
Course MMC-31
(Discrete Mathematics)

Time: Three Hours Full Marks: 40

Questions are of value as indicated in the margin
Notations and symbols have their usual meaning

Answer *any four* questions

1. (a) Define well-formed formula in classical propositional logic. When an argument form is said to be valid? Construct a formal proof of validity of the argument

$$(C \to D) \wedge (E \to F)$$
$$G \to (C \vee E)$$
$$\therefore\ G \to (D \vee F).$$

[3]

(b) Examine the validity of the argument

$$I \vee (J \wedge K)$$
$$(I \vee J) \to (L \leftrightarrow M)$$
$$(L \to \sim M) \to (M \wedge \sim N)$$
$$(N \to O) \wedge (O \to M)$$
$$(J \to K) \to O$$
$$\therefore O.$$

[3]

(c) Construct a formal proof of validity for each of the arguments in the following:

(i) If either Saudi Arabia raises the price of oil or Saudi Arabia does not raise the price of oil, then India will be in grave difficulties. Therefore, India will be in grave difficulties.

(ii)
$$P \to (Q \wedge R)$$
$$(Q \vee S) \to T$$
$$P \vee S$$
$$\therefore T$$

[2+2]

2. Construct formal proofs of validity for the argument forms:

(a) $(\forall x)[(B(x) \to C(x)) \wedge (D(x) \to E(x))]$
$(\forall x)[(C(x) \vee E(x)) \to \{[F(x) \to (G(x) \to F(x)] \to (B(x) \wedge D(x))\}]$.
$\therefore (\forall x)(B(x) \leftrightarrow D(x))$. [4]

(b) Doctors and lawyers are college graduates. Any altruist is an idealist. Some lawyers are not idealist. Some doctors are altruists. Therefore, some college graduates are idealists. [4]

(c) $(\forall x)(\exists y)(K(x) \wedge L(y))$
$\therefore (\forall x)K(x) \wedge (\exists y)L(y)$. [2]

3. (a) Show that a $k-$regular graph with girth four has at least $2k$ vertices, with equality only for the complete bipartite graph $K_{k,k}$. [4]

(b) Show that every $n-$vertex graph with at least n edges contains a cycle. [3]

(c) Define radius ($rad(G)$) and diameter ($diam(G)$) of a graph G. Show that for any graph G

$$rad(G) \le diam(G) \le 2\ rad(G).$$

[3]

4. (a) Define Euler graph. Show that if W is a walk from vertex u to vertex v, then W contains an odd number of $(u-v)$ paths. Hence show that a connected graph is Eulerian if and only if each of its edges lies on an odd number of cycles. [5]

(b) Define spanning tree. Write Breadth First Search algorithm for finding a spanning tree in a graph G. Illustrate the same with an example of your choice. [5]

5. (a) State and prove Erdös-Szekeres theorem on counting large increasing/decreasing number subsequences. Illustrate the same with a sequence of your choice. [3+2]

(b) State the Inclusion-Exclusion principle and use it to find the number of derangements among the permutations of $\{1, 2, \cdots n\}$. [2+3]

6. (a) Count the number of paths of length n in the $xy-$ plane starting from $(0,0)$ with steps

$$R : (x, y) \to (x+1, y),$$
$$L : (x, y) \to (x-1, y),$$
$$R : (x, y) \to (x, y+1).$$

It is given that a step $R(L)$ is not to be followed by a step $L(R)$. [5]

(b) State and prove Burnside's theorem on counting the number of equivalence classes. [5]

Use separate answer script for each unit

M.Sc. Examination - 2022

Semester-III
Mathematics
Course: MMC-32

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin
Notations and symbols have their usual meanings

Unit-I [Advanced Mathematical Statistics - I (Marks: 20)]

Answer *any two* questions

1. Describe the one-sample non-parametric Wilcoxon signed-rank hypothesis test procedure. The systolic blood pressures (mmHg) of 13 patients undergoing a drug therapy for hypertension are as given below

Patients	1	2	3	4	5	6	7	8	9	10	11	12	13
SBP	183	178	152	157	194	163	144	114	179	150	118	158	165

Can we conclude on the basis of these data and the said procedure that the median systolic blood pressure is less than 165 mmHg? Assume $\alpha = 0.05$, the significance level of the test. [5+5]

2. (a) In Normal regression analysis, find the distributions of the estimators $\hat{\beta}$ and $\hat{\alpha}$ which are unbiased estimators of β and α, the regression co-efficients. [6]

(b) Show that an unbiased estimator S^2 of σ^2 is given by

$$S^2 = \frac{n\hat{\sigma}^2}{n-2};$$

where $\hat{\sigma}$, the maximum likelihood estimator of σ, is to be determined by you. [4]

3. Consider the standard model of one-way ANOVA given by

$$Y_{ij} = \mu_i + \epsilon_{ij} \quad \text{for } i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n$$

where $m \geq 2$ and $n \geq 2$. Assume that each random variable

$$Y_{ij} \sim N(\mu_i, \sigma^2).$$

Compute the MLE's of the parameters $\mu_i (i = 1, 2, \ldots, m)$ and σ^2. Show also that under the null hypothesis $H_0: \mu_1 = \mu_2 = \ldots = \mu_m = \mu$(,say)

i) $$\frac{SS_W}{\sigma^2} \sim \chi^2(m(n-1));$$

ii) $$\frac{SS_B}{\sigma^2} \sim \chi^2(m-1);$$

iii) $$\frac{SS_T}{\sigma^2} \sim \chi^2(nm-1).$$

[4+6]

Unit-II [Theory of Chaos(Marks: 20)]

Answer *any two* questions.

1. (a) Convert the following equations in polar coordinate and explain the type of bifurcation arises

$$\frac{dx}{dt} = \mu x - y + x(x^2+y^2) - x(x^2+y^2)^2$$
$$\frac{dy}{dt} = x + \mu y + y(x^2+y^2) - y(x^2+y^2)^2$$

[3]

(b) Explain one-dimensional saddle-node bifurcation with suitable diagrams. Find the conditions under which the system $\dot{x} = f(x, \mu)$, (x, and μ are real numbers, μ is the parameter), undergoes a saddle-node bifurcation . Obtain the normal form of saddle-node bifurcation. [3+3+1]

2. (a) Define SDIC, topological transitivity and topological mixing of a map $f : \mathbb{R} \to \mathbb{R}$. Show that the doubling map attains SDIC and topological transitivity on a unit circle. [3+3]

(b) Define Lyapunov exponent. Show that the Lyapunov exponent of an one-dimensional map $x_{n+1} = f(x_n)$ can be expressed as $\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} ln|f'(x_i)|$.Obtain the Lyapunov exponent of the tent map, $T : [0, 1] \to [0, 1]$. [1+2+1]

3. (a) What do you mean by a similarity transformation? Define translation, rotation and scaling operations with their physical explanation. [1+2]

(b) Define self similarity in space, self similarity in time and statistical self similarity. [2]

(c) Show that the Cantor set has zero measure.What is its similarity dimension. [2]

(d) How is the Mendelbort set constructed? What are escape set and prisoner set? When is a Julia set become attractor or repeller ? [1+1+1]

M.Sc. Examination - 2022

Semester-III

Mathematics

Course: MMC-33

(Fluid Mechanics)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin
Notations and symbols have their usual meanings

Answer ***any four*** questions

1. (a) Show that Eulerian and Lagrangian forms of equations of continuity are equivalent. [3]

   (b) Determine the acceleration at the point $(2, 1, 3)$ at $t = 0.5$ sec, if $u = yz + t, v = xz - t$ and $w = xy$. [3]

   (c) If for a two-dimensional flow the velocities at a point in a fluid are expressed in the Eulerian coordinates by $u = x + y + 2t$ and $v = 2y + t$, determine the Lagrange coordinates as functions of the initial positions x_0 and y_0 and the time t. [4]

2. (a) Establish the relation between the angular velocity vector and vorticity vector. [3]

   (b) Define stream line, path line and vortex line. Find their differential equations in Cartesian and cylindrical polar co-ordinates. [3]

   (c) At a point in an incompressible fluid having spherical polar coordinates (r, θ, ϕ) the velocity components are $[2Mr^{-3}cos\theta, Mr^{-3}sin\theta, 0]$, where M is a constant. Show that the velocity is of the potential kind. Find the velocity potential and the equation of the stream lines. [2+1+1]

3. (a) Find $f(r)$ if $\phi = xf(r)$ be a possible form for the velocity potential of an incompressible liquid motion. Hence show that the surfaces of constant speed are $(r^2 + 3x^2)r^{-8}$ = constant if it is given that the liquid speed $q \to 0$ as $r \to \infty$. [6]

   (b) A steady inviscid incompressible fluid flow has a velocity field $u = fx, v = -fy, w = 0$, where f is a constant. Derive an expression for the pressure field $p(x, y, z)$ if the pressure at the origin, $p(0, 0, 0) = p_0$ and external force $\vec{F} = -g\hat{k}z$. [4]

4. (a) State and prove the Milne-Thomson Circle theorem. Using this theorem determine the image system for a source outside a circle. [2+3+2]

   (b) Show that the mean value of ϕ over any spherical surface, throughout whose interior $\nabla^2\phi = 0$, is equal to the value of ϕ at the center of the sphere. [3]

5. (a) State and prove the Kelvin's minimum energy theorem. [2+3]

   (b) A source and a sink of equal strength are placed at the points $(\pm a/2, 0)$ within a fixed circular boundary $x^2 + y^2 = a^2$. Show that the streamlines are given by $\left(r^2 - \frac{a^2}{4}\right)(r^2 - 4a^2) - 4a^2y^2 = ky\left(r^2 - a^2\right)$. [5]

6. (a) Find the complex potential for the flow past a fixed circular cylinder without assuming the complex potential for the motion of a circular cylinder moving in an infinite mass of the liquid at rest at infinity with velocity U in the direction $x-$axis. [5]

   (b) Find the velocity distribution in the generalized Coutte flow. Determine the shearing stress, skin friction and the coefficient of friction at both the walls. [5]

Use separate answer script for each unit

M.Sc. Examination, 2022

Semester-III
Mathematics
Course: MMC-34 (New Syllabus)
(Calculus of Variations and Special Functions)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 20)
(Calculus of Variations)
Answer *any two* questions.

1. (a) Find the extremal of the functional $I[y(x)] = \int_{x_0}^{x_1} \left(y + \frac{y^3}{3}\right) dx$ subject to the boundary conditions $y(x_0) = y_0, y(x_1) = y_1$. [2]

   (b) Find the extremal of the functional $V[y(x)] = \int_a^b (y''^2 - 2y'^2 + y^2 - 2y \sin x) dx$. [4]

   (c) Show that the necessary condition for existence of extremals of the functional
   $$I[z(x,y)] = \iint_\Omega F(x, y, z, z_x, z_y) dx dy$$
   is $F_z - \frac{\partial}{\partial x}(F_{z_x}) - \frac{\partial}{\partial y}(F_{z_y}) = 0$, where $z(x, y)$ is continuous and has continuous second order partial derivatives in Ω and is prescribed on the boundary $\partial\Omega$. [4]

2. (a) Show that for the functional of the form $J[y] = \int_a^b f(x, y)\sqrt{1 + y'^2}\, e^{\pm \tan^{-1} y'} dx$, the transversality conditions reduce to the requirement that the curve $y = y(x)$ intersect the curves $y = \phi(x)$ and $y = \psi(x)$ (along which its end points vary) at an angle of 45°. [5]

   (b) Find the shortest distance between the parabola $y = x^2$ and the straight line $y = x - 5$. [5]

3. (a) Find the extremal of the functional $I[y(x)] = \int_0^\pi (y'^2 - y^2) dx, y(0) = 0, y(\pi) = 1$. subject to $J[y(x)] = \int_0^\pi y dx = 1$. [4]

   (b) Define central field with a suitable example. [2]

   (c) Using Legendre condition, test for an extremum of the functional $J[y(x)] = \int_0^2 (4 - e^{y'})\, dx$, subject to the boundary conditions determined from the relation $y(x) = \frac{x}{2}$. [4]

Unit-II (Full Marks: 20)
(Special Functions)
Answer *any two* questions.

1. (a) Using power series method, find the indicial equations of Hermite and Legendre equations. [4]

   (b) Express $x^3 + x^2 - 3x + 2$ in terms of Laguerre polynomials. [4]

   (c) Write down the expression for $J_n(x)$. Also find the relation between $J_n(x)$ and $J_{-n}(x)$.(n being an integer). [1+1]

2. (a) Prove that $P_{2m}(0) = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2}$. [3]

   (b) Express $J_4(x)$ in term of J_0 and J_1. [3]

   (c) Prove that $e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$ [4]

3. (a) Prove that (*i*) $J_{-1/2}(x) = \sqrt{(\frac{2}{\pi x})}\cos(x)$, (*ii*) $J_{1/2}(x) = \sqrt{(\frac{2}{\pi x})}\sin(x)$, (*iii*) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}$. [2+2+1]

   (b) Prove that (*i*) $\int_{-1}^{1} P_m(x)P_n(x)\, dx = 0$ if $m \neq n$, (*ii*) $\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$ if $m = n$. [2+3]

Use separate answer script for each unit

M. Sc. Examination-2022

Semester-III
Mathematics
Elective Course: MMC-35
(Galois Theory -I and Multivariable Analysis)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit I: Galois Theory I (Full Marks: 20)

Answer ***any four*** questions.

1. (a) Show that every field of characteristic zero is an extension of $\mathbb{Q}$. [3]
   (b) Show that the extension $\mathbb{C}/\mathbb{R}$ has no proper nontrivial intermediate fields. [2]
2. (a) Let F/K be a field extension, $c \in F$ be algebraic over K and $m(x)$ be the minimal polynomial of c over K. Show that $[K(c):K] = \deg m(x)$. [3]
   (b) Let F/K be a field extension and $c \in F$. If $c \in F$ is algebraic of odd degree over K, then show that $K(c) = K(c^2)$. [2]
3. (a) Let $a, b \in \mathbb{C}$ be two algebraic numbers of degrees 3 and 5 respectively. Find $[\mathbb{Q}(a,b):\mathbb{Q}]$. [3]
   (b) Let F/K be a field extension. If it has only finite number of intermediate fields, show that F/K is a finite extension. [2]
4. (a) Let $f(x) \in K[x]$ be a nonconstant polynomial. Prove that an extension S of K is a splitting field of $f(x)$ if and only if $f(x)$ splits over S, but over no proper intermediate fields of S/K. [3]
   (b) Show that there is an isomorphism $\sigma : \mathbb{Q}(\sqrt[3]{2}) \longrightarrow \mathbb{Q}(\sqrt[3]{2}\omega)$ such that $\sigma(\sqrt[3]{2}) = \sqrt[3]{2}\omega$ and $\sigma(q) = q$ for every $q \in \mathbb{Q}$. [2]
5. (a) Find a splitting field S of $x^2 + x + 1$ over $\mathbb{Z}_2$. Also find $[S:\mathbb{Z}_2]$. [2]
   (b) Let F be a field extension of $\mathbb{R}$. If $a \in F$ is algebraic over $\mathbb{R}$, then show that $a \in \mathbb{R}$ or $\mathbb{R}(a) \simeq \mathbb{C}$. [3]
6. (a) Show that for every prime p and $n \in \mathbb{N}$, there is a field F of order p^n. [3]
   (b) For any finite field F of order n, show that there is a polynomial $f(x) \in F[x]$ of degree n having no roots in F. [2]

Unit-II : Multivariable Analysis (Full Marks: 20)

Answer any two questions

7. (a) Give definition of differentiability of a function $f : \mathbb{R}^n \to \mathbb{R}^m$. Show that $f = (f_1, ..., f_m)$ is differentiable at x if and only if $f_1, ..., f_m$ are differentiable at x. [3]
   (b) If f is a C^1 function then prove that f is differentiable. [4]
   (c) If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ are differentiable functions then show that $f \circ g$ is also differentiable. [3]
8. (a) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function. Prove that $D_iD_jf = D_jD_if$. [3]
   (b) Define directional derivative $D_vf(a)$ of the function $f : \mathbb{R}^n \to \mathbb{R}^m$ in the direction v. State and prove Taylor's theorem for a C^{k+1} function $f : \mathbb{R}^n \to \mathbb{R}$. [1+4]
   (c) Find kth degree Taylor's polynomial of $e^{x_1+\dots+x_n}$. [2]
9. (a) Define the quadratic form $q(h)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a critical point a. Show that f has neither a maximum nor a minimum at the point a if $q(h)$ is non definite. [4]
   (b) If a quadratic form q attains an extremum at $v \in S^{n-1}$ then show that v is an eigen vector of the linear mapping associated with q. [4]
   (c) Let $h(x, y, z) = (x, 3xy - z, 4y - x^2y)$. Show that h^{-1} exists in the domain where $x \neq \pm 2$. [2]

M. Sc. Examination-2022

Semester-III

Mathematics

MM0-31(A7/P8)(New)

(Lie Theory of Ordinary and Partial Differential Equations)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Define Lie group of transformations in its infinitesimal form and introduce infinitesimal generator for a Lie group of transformations. Verify whether the set of transformations $(x,y) \to (x^*, y^*) = \mathcal{T}_\epsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ is a Lie group of transformations. Find the infinitesimal generator, if yes. [2+5+3]

2. What do you mean by the invariant function for a Lie group of point transformations in some geometric space. State and prove the necessary and sufficient condition for a function to be an invariant function for a given Lie group of transformations. Find invariant function for the Lie group of point transformations $(x,y) \to (x^*, y^*) = (e^\epsilon x, e^{2\epsilon} x)$. [2+5+3]

3. Define tangency or contact condition among the variables in the geometric space. Use tangency condition to prolong the Lie group of point transformations $(x,y) \to (x^*, y^*) = (e^\epsilon x, e^{2\epsilon} y)$ to the jet space $G^{(2)}$, whose points are described by the symbol (x, y, y', y''). Prove the formula $y^{*(k)} = \frac{d^k y^*}{dx^{*k}} = \frac{D[Y^{(k-1)}(x,y,\cdots,y^{(k-1)},\epsilon)]}{D[X(x,y,\epsilon)]}$. [2+6+2]

4. Find equation for $\phi(x)$ so that $\hat{X} = \phi(x)\frac{\partial}{\partial y}$ will be an infinitesimal generator for the Lie group of point transformations admitted by the ordinary differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x),$$

where $p(x), q(x), g(x) \in \mathcal{C}(\mathbb{R})$. Find other infinitesimal generator, if exists. Obtain canonical variables for the Lie group of symmetry transformations generated by $\hat{X}$ mentioned above. [5+2+3]

5. Show that the heat equation $u_t = u_{xx}$ admits a Lie group of symmetry transformations generated by the infinitesimal generators $\hat{X} = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} - \frac{1}{2}(t + \frac{1}{2}x^2)u\frac{\partial}{\partial u}$. Transform the heat equation to an ordinary differential equation by using similarity variables for $\hat{X}$. [6+4]

6. Examine whether the KdV equation $u_t + 6uu_x + u_{xxx}$ admits a Lie group of symmetry transformations generated by the infinitesimal generators $\hat{X}^{KdV} = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$. Transform the KdV equation to an ordinary differential equation by using similarity variables for $\hat{X}^{KdV}$. Find a group invariant solution of the KdV equation using some boundary condition to be specified by you. [3+4+3]

M. Sc. Examination-2022
Semester-III
Mathematics
Optional Course : MMO-31 (P03)
(Advanced Real Analysis-I)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer **question no.6** and any **three** from the rest

1. (a) Show that the set of all points of discontinuities of any function is a F_σ-set. [3]
   (b) Define lim sup of a function f at a point a. If one of the Dini derivates of a continuous function f is continuous at x then show that $f'(x)$ exists. [2+4]
   (c) State and prove Jensen inequality for convex functions. [3]
2. (a) State and prove Zygmond theorem. [4]
   (b) Define Cantor function on $[0,1]$ and show that it is continuous and non decreasing on $[0,1]$. [2+3+3]
3. (a) Show that if a function f is absolutely continuous on $[a,b]$ then f is of bounded variation, continuous and satisfies Lusin condition on [a,b] [2+3+4]
   (b) Show that for any set E, the set of all isolated points is countable [3]
4. (a) What is Vitali cover. State Vitali's covering theorem [1+2]
   (b) Suppose f is Lebesgue integrable on $[a,b]$ and let $F(x) = \int_a^x f$ for all $x \in [a,b]$. Show that $F' = f$ a.e on $[a,b]$ [5]
   (c) Show that a function f is *usc* on $[a,b]$ if and only if the set $\{x \in [a,b] : f(x) \geq c\}$ is closed for every real number c. [4]
5. (a) Let a sequence of Baire class 1 functions $\{f_n\}$ converges uniformly to f. Show that f is also a Baire class 1 function. [5]
   (b) Let $f : [a,b] \to \mathbb{R}$. If $f = 0$ a.e on $[a,b]$, then show that f is Henstock integrable on $[a,b]$ and $\int_a^b f = 0$ [4]
   (c) State and prove the three chords Lemma for convex functions. [3]
6. Answer any two
   (a) Give example of a Baire 2 function which is not Baire 1. [2]
   (b) Show that (*i*) Cantor functions (*ii*) Monotone functions are Baire 1 functions [2]
   (c) Find Dini derivates at the point zero of the function $f(x) = |x| \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ [2]

M. Sc. Examination-2022

Semester-III

Mathematics

Course: MMO-31 (P-02)

(Advanced Functional Analysis-I)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer any four questions

1. (a) Define a topological vector space (tvs). For a fixed $a \in X$ and $\lambda(\neq 0) \in \Phi$, define $T_a, M_\lambda : X \to X$ by $T_a(x) = x + a$ and $M_\lambda(x) = \lambda x,\ x \in X$. Show that T_a and M_λ are homeomorphism. [6]

   (b) Let X be a tvs. Then prove the following:

   (i) Every neighborhood of θ in X contains an absorbing neighborhood of θ.

   (ii) If B is a balanced subset of X then $\overline{B}$ is also so. [4]

2. (a) When is a tvs X said to be locally compact? If X is a tvs and Y is locally compact subspace of X then show that Y is closed. [5]

   (b) Let X be a tvs and E be a subset of X. Prove that following statements are equivalent:

   (i) E is bounded.

   (ii) If $\{x_n\}$ is any sequence in E and $\{\alpha_n\}$ is any sequence of scalars such that $\lim_{n\to\infty} \alpha_n = 0$ then $\lim_{n\to\infty} \alpha_n x_n = \theta$. [5]

3. (a) Let X be a linear space and A be a convex and absorbing set in X containing θ. If μ_A be the Minkowski functional defined on A then prove the following:

   (i) $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$ for all $x, y \in X$.

   (ii) $\mu_A(tx) = t\mu_A(x)$ for all $t \geq 0$ and $x \in X$.

   (iii) μ_A is a seminorm if A is a balanced set in X.

   (iv) If $B = \{x \in X : \mu_A(x) < 1\}$ and $C = \{x \in X : \mu_A(x) \leq 1\}$ then $B \subset A \subset C$ and $\mu_A = \mu_B = \mu_C$. [6]

   (b) If Y is a subspace of a tvs X and Y is an F-space (in respect to the topology inherited from X), then prove that Y is a closed subspace of X. [4]

4. (a) State and prove Hahn-Banach theorem for a complex linear space (assuming that the corresponding result holds for a linear space). [4]

   (b) Let A and B be two nonempty disjoint convex sets in a tvs X. If A is open then show that there exists $\Lambda \in X^*$ (dual space of X) and $\gamma \in \mathbb{R}$ such that $Rl\Lambda(x) < \gamma \leq Rl\Lambda(y)$ for every $x \in A$ and $y \in B$. [6]

5. (a) Let X and Y be two tvs and $\{\Lambda_n\}$ be a sequence of continuous linear mappings of X into Y. Let $C = \{x \in X : \{\Lambda_n(x)\}$ *is a Cauchy sequence in* $X\}$ and $L = \{x \in X : \{\Lambda_n(x)\}$ *is convergent in* $Y\}$. Prove that,

(i) if C is of 2nd Category in X then $C = X$.

(ii) if L is of 2nd Category in X and Y is an F-space then $L = X$.

(b) Prove that every locally compact tvs is finite dimensional. [4+1]

(c) If C is a convex set in a tvs X, then prove that C° and $\overline{C}$ are convex sets in X. [3]

6. (a) State and prove Banach-Alagulu theorem. [2]

(b) Let X be a tvs. If X is metrizable by an invariant metric d then show that $d(nx, \theta) \leq nd(x, \theta)$, n being a positive integer. [6]

(c) For any seminorm p on a vector space X, prove that the set $\{x \in X : p(x) \leq 1\}$ is convex and balanced. [2]

[2]

M. Sc. Examination - 2022

Semester-III

Mathematics

Paper: MMO 31 (P01)

(Advanced Complex Analysis-I)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meaning.

Answer ***any four*** questions.

1. (a) Prove that if $f(z) = u(x, y) - iv(x, y)$ is analytic in a domain D, then $v(x, y)$ is a harmonic function in D. [2]

   (b) When a real or complex valued function f, defined in a domain D is said to have the mean value property in D ? Prove that any analytic function defined in a domain D has the mean value property in D. Is it true for any harmonic function defined in D ? Justify your answer. [5]

   (c) State and prove Poisson's integral formula. [3]

2. (a) Let f is an entire function such that $|f(z)| \leq 10log|z|$ for each z with $|z| \geq 2$. Then prove that $f(z)$ be a constant. [2]

   (b) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function then prove that $|a_n|r^n + 2\Re f(0) \leq max\{4A(r), 0\}$ for all values of $n > 0$ and $r > 0$. [4]

   (c) State and prove Hadamard's three circles theorem. [4]

3. (a) Let $f(z)$ be a non-constant analytic function in $|z| \leq R$. Then show that for $0 \leq r < R$, $M(r) < \frac{2R}{R-r}\{A(R) + |f(0)|\}$. [3]

   (b) If $f(z)$ is an integral function with $f(0) \neq 0$, then show that $f(z) = f(0)G(z)e^{g(z)}$, where $G(z)$ is a product of primary factors and $g(z)$ is an integral function. [2]

   (c) Define order of an integral function. Find the order of $e^{P(z)}$, where $P(z)$ is a polynomial of degree m. [5]

4. (a) If $f(z)$ is an entire function of order ρ and $P(z)$ is any non-zero polynomial such that $\frac{f(z)}{P(z)}$ is an entire function, then show that $\frac{f(z)}{P(z)}$ is also of order ρ. [2]

   (b) If $f(z)$ is analytic in $|z| \leq R$, then show that for $0 \leq r < R$
   $\frac{M(r)-|f(0)|}{r} \leq M'(r) \leq \frac{M(R)}{R-r}$. [3]

   (c) If $f(z)$ is an entire function of order ρ, then show that for every $\epsilon(> 0)$, the inequality $n(r) \leq r^{\rho+\epsilon}$ holds for all sufficiently large r, where $n(r)$ denotes the number of zeros of $f(z)$ in the closed disc $|z| \leq r$. [5]

5. (a) Show that the exponent of convergence of zeros of $\cos 3z$ is 1. Hence or otherwise find the order of $\cos 3z$. [4]

   (b) If $f(z)$ is an integral function of finite order ρ and $r_1, r_2, ...r_n, ...$ are the moduli of the zeros of $f(z)$ then show that $\sum_{n=1}^{\infty} \frac{1}{r_n^\alpha}$ converges if $\alpha > \rho$. [3]

   (c) Define canonical product for an entire function $f(z)$. If ρ_1 is the exponent of convergence of zeros of $f(z)$ and if P is the genus, then show that
   $\rho_1 - 1 \leq P < \rho_1$. [3]

6. (a) Define type of an integral function. Find the type of $\sum_{n=1}^{\infty} \frac{z^n}{(n!)^3}$. [3]

   (b) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of finite non-zero order ρ, then find the expression for ρ in terms of a_n. [5]

   (c) Give an example of an integral function having order $\alpha(> 0)$ and type $\frac{1}{2\alpha}$. [2]

Use separate answer script for each unit

M.A./M.Sc. Examination-2023

Semester-III
Mathematics
MMO 31 (A10)
(Nonlinear Differential Equation-I)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Nonlinear Differential Equation (Marks: 40)
Answer ***any four*** questions.

1. (a) Using the travelling wave transformation find the soliton solution of KP Equation, $(u_t+6uu_x+u_{xxx})_x+3u_{yy}=0$. [5]

   (b) Using the travelling wave transformation find the soliton solution of mKP Equation, $(u_t+3u^2u_x+u_{xxx})_x+u_{yy}=0$ [5]

2. What is meant by conservation principle and integrals of motion of an evolution equation? Hence Obtain the approximate analytic solution of damped KdV equation and forced KdV equation. [2+4+4]

3. Explain the tanh method, hence find the Shock wave solution of the Burger equation, $u_t+uu_x+u_{xx}=0$. [10]

4. Explain Cole-Hopf transformation. Hence solve the following initial value problem

$$\begin{aligned} u_t+uu_x-\epsilon u_{xx} &= 0, \qquad -\infty<x<\infty, \\ u(x,0) &= f(x). \end{aligned}$$

[10]

5. (a) Using the sine-cosine method, find the soliton solution of Boussinesq Equation , $u_{tt}-au_{xx}+3(u^2)_{xx}-bu_{xxxx}=0$ [5]

   (b) Using the sine-cosine method, find the soliton solution of RLW Equation, $u_t+au_x-6uu_x-bu_{txx}=0$. [5]

6. If $u=u_0Sech^2(\frac{x-ct}{\delta})$ will be a Soliton solutin of the KdV Equation $u_t+6uu_x+u_{xxx}=0$, then fine the relation among u_o, v and δ. Also explain it physically. [10]

M. Sc. Examination, 2022

Semester-III
Mathematics
Optional Course : MMO-31 (A9) (New)
(Mathematical Pharmacology-I)

Time: Three Hours   Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Obtain the transient concentration of receptor/ligand complex ($c(t)$) for simple monovalent cell surface binding model in case of constant ligand concentration. Hence obtain the equilibrium concentration. [7+3]

2. Define specific and non-specific binding. Prove that the free ligand concentration is diminished by non-specific binding and the total binding is a multiple of specific binding. [3+4+3]

3. Write down the model equations that exhibit interconversions of receptor and complex states with varying rate constants. Assuming equilibrium binding of ligand to both receptor forms, obtain the unsteady solutions for complexes. [4+6]

4. Write down the model equations for endocytosis. Obtain the ratio of total number of surface receptor in the presence of ligand to that in the absence of ligand. Discuss the case when endocytic downregulation is predicted. [4+4+2]

5. Formulate a single endosome model in which bivalent ligands bind to monovalent receptors. Write down the descriptions of variables and parameters involved. Nondimensionalise the equations. [4+2+4]

6. Derive Fick's first law of diffusion. Suppose a bolus of N drug molecules is injected into a long cylinder at t=0 so that all the molecules are present within an infinitesimal volume at x=0 and applying sink condition at $x \to \pm\infty$, obtain the expression for concentration at time t. [5+5]

M. Sc. Examination 2022
Semester-III
Mathematics
Course: MMO-31 (A08) (New Syllabus)
(Magnetohydrodynamics-I)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) State under what conditions an ionized gas is said to be a plasma. [2]
   (b) Discuss the motion of a charged particle in presence of a uniform magnetic field. Hence show that the particle describes a helical path. [3+2]
   (c) Prove that the energy of a charged particle under uniform electric and magnetic fields is conserved. [3]

2. (a) Find the one-dimensional Boltzmann distribution for a non-relativistic classical charged particle in thermodynamic equilibrium. [3]
   (b) Define magnetic moment of a charged particle. Show that it remains invariant throughout the motion under the magnetic field. Hence discuss the concept of a magnetic mirror. [3+2+2]

3. (a) Define the phase velocity and group velocity of a plane propagating wave. [2+2]
   (b) Obtain the dispersion relation for electrostatic ion-cyclotron waves in a magnetoplasma with inertialess thermal electrons and inertial cold positive ions. Distinguish between the characteristics of these waves and those of electrostatic ion-acoustic waves. [4+2]

4. For the propagation of high-frequency electromagnetic waves along a constant magnetic field, obtain expressions for refractive indices of the left- and right-circularly polarized waves in a magnetoplasma. Hence find the cut-off and resonant (if any) frequencies of these waves. [8+2]

5. Write short notes on (i) Upper-hybrid frequency (ii) Lower-hybrid frequency. [5+5]

6. Explain the phenomenon of Landau damping of electron plasma oscillations in a collisionless unmagnetized plasma. Using the kinetic Vlasov-Poisson system of equations, obtain expressions for the plasma dielectric function of wave dispersion and the Landau damping rate for a given background distribution of electrons. [2+8]

M. Sc. Examination-2022

Semester-III

Mathematics

MMO 31 : Optional Paper

Rings and Modules-I

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Let V be an inner product space and $T : V \longrightarrow V$ be a linear operator. Then prove that $\chi_T(t)$ splits into linear factors if and only if there exists an orthogonal basis β of V such that $[T]_\beta$ is upper triangular. [3]

   (b) Let V be a complex inner product space. Prove that $T : V \longrightarrow V$ is normal if and only if V has an orthonormal basis of eigen vectors of T. [4]

   (c) Let V be a complex inner product space. Show that $T : V \longrightarrow V$ is normal if and only if $T^* = g(T)$ for some polynomial $g(t)$. [3]

2. (a) Let $A \in M_n(\mathbb{C})$. Prove that A is Hermitian if and only if X^*AX is real for all $X \in \mathbb{C}^n$. [3]

   (b) Let $A \in M_n(\mathbb{C})$. Show that there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that U^*AU is an upper triangular matrix. [4]

   (c) Prove that a matrix $A \in M_n(\mathbb{C})$ is Hermitian if and only if A has n real eigen values and $\mathbb{C}^n$ has an orthonormal basis of eigen vectors of A. [3]

3. (a) Let $A \in M_n(\mathbb{C})$ be a positive semidefinite matrix and $X \in \mathbb{C}^n$. Show that $X^*AX = 0$ if and only if $AX = 0$. [3]

   (b) Let G be the gram matrix of $\{v_1, v_2, \cdots, v_n\}$. Show that G is Hermitian and positive semidefinite. Also show that $rank(G) = \dim span\{v_1, v_2, \cdots, v_n\}$. [4]

   (c) Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix. Show that A is positive definite if and only if every eigen value of A is positive. [3]

4. (a) Let $A \in M_n(\mathbb{C})$ be a positive semidefinite matrix. Prove that there exists a unique matrix B such that $B^2 = A$. [4]

   (b) Let $B \in M_{m\times n}(\mathbb{C})$ and $A = B^*B$. Show that A is positive definite if and only if B is of full column rank. [3]

   (c) Let V be a vector space of dimension n. Show that the dimension of $\mathcal{B}(V)$ is n^2. [3]

5. (a) Let f be a bilinear form on a vector space V. Show that $r(f) = r(L_f)$. [3]

   (b) Show that a reflexive bilinear form is either symmetric or alternating. [4]

   (c) Let f be a nondegenerate reflexive bilinear form. Then show that for every $\phi \in V^*$, there exists a unique $v \in V$ such that $\phi(x) = f(x, v)$. [3]

6. (a) State the principal axis theorem. Find the principal axis form of the quadratic from $Q(X) = 4(xy + yz + zx)$. [3]

   (b) Prove that two real symmetric matrices A and B are congruent if and only if they have the same inertia. [4]

   (c) Find the inertia and type of the surface $x^2 + 2y^2 + 3z^2 - 4xy + 6yz - 8zx = 1$. [3]

M.Sc. Examination - 2022
Semester-III
Mathematics
Course: MMO-31 (A-03)
(Computational Fluid Dynamics - I)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin
Notations and symbols have their usual meanings

Answer ***any four*** questions

1. (a) Find the fourth order accurate finite difference analog of first order partial derivative using the values of the dependant variable at the $(i \pm 2, j)$ and $(i \pm 1, j)$ points. [4]

   (b) Show that

   i. $\dfrac{\partial u_{ij}}{\partial x} = \dfrac{1}{2h}\dfrac{\delta_x u_{ij}}{1+\delta_x^2/6} + o(h^4)$ [3]

   ii. $\dfrac{\partial^2 u_{ij}}{\partial x^2} = \dfrac{1}{h^2}\dfrac{\delta_x^2 u_{ij}}{1+\delta_x^2/12} + o(h^4)$ [3]

2. If the heat conduction equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ is approximated by the schemes of Schmidt, Laasonen and Crank-Nicholson method then determine the truncation errors. Also, investigate the stability of Crank-Nicholson method using Von-Neuman method. [10]

3. A difference method for the numerical solution of
   $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, t > 0$
   under suitable initial and boundary conditions is written as
   $u_m^{n+1} = \dfrac{4}{3}u_m^n - \dfrac{1}{3}u_m^{n-1} + \dfrac{2\lambda}{3}\delta_x^2 u_m^{n+1}$
   where the parameter $\lambda = k/h^2$. If the initial condition is

   $$u(x,0) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & 1/2 \le x \le 1 \end{cases}$$

   and the boundary conditions are
   $u(0,t) = u(1,t) = 0$,
   find the finite difference solution when the grid size $h = 1/4$ and $\lambda = 1/2$. Integrate upto two time steps. [10]

4. Derive Wendroff, Lax-Wendroff and Leap-Frog schemes for the solution of first order hyperbolic partial differential equation. Also, find the conditions of stability and truncation errors involved. [10]

5. Find the solution of the initial value problem
   $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0$
   together with an initial condition

   $$u = \begin{cases} 0, & x < 0 \\ x/2, & 0 \le x \le 2 \\ 2-(x/2), & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

   using (i) Lax-Wendroff scheme and (ii) Leap-frog scheme with $h = 0.5, r = 0.5$ where $r = k/h$. Integrate upto two time steps. [10]

6. Solve the mixed boundary value problem

$\nabla^2 u = 0, 0 \leq x, y \leq 1$

$$u = \begin{cases} 2x, & y = 0, 0 \leq x \leq 1 \\ 2x - 1, & y = 1, 0 \leq x \leq 1 \end{cases}$$

$u_x + u = 2 - y, x = 0, \quad 0 \leq y \leq 1$

$u = 2 - y, x = 1, \quad 0 \leq y \leq 1.$

Use the five point formula with $h = k = 1/3$. [10]

M. Sc. Examination-2022

Semester-III

Mathematics

Course: MMO-31(A04)

Differential Equations in Ecology

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any ten*** questions.

1. Consider the "rabbits vs sheep" problem

$$\frac{dx}{dt} = x(3 - x - y).$$
$$\frac{dy}{dt} = y(2 - x - y),$$

where $x, y \geq 0$. Draw the nullclines, find the equilibrium points and investigate their stability. [4]

2. Consider a linear birth-death-immigration-emigration process in a stochastic model. Derive a stochastic differential equation for p_n, where p_n is the probability of n individuals in the system at any time t. [4]

3. What are the advantages of a dimensionless model? Using suitable transformation, transform the system

$$\frac{dx}{dt} = ax - bxy.$$
$$\frac{dy}{dt} = -cy + dxy\frac{y}{N + y}.$$

into the system

$$\frac{du}{dt} = u - uv.$$
$$\frac{dv}{dt} = -\gamma v + \frac{uv^2}{1 + \xi v}.$$

[4]

4. Consider the differential equation $\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - hx$. Sketch the bifurcation diagram by considering the harvesting parameter h as the bifurcation parameter. [4]

5. Prove that the system

$$\frac{dx}{dt} = rx - bx^2 - \frac{pxy}{q + x},$$
$$\frac{dy}{dt} = \frac{pxy}{q + x} - \frac{rxy}{1 + \alpha x^2} - d_2 y.$$
$$\frac{dz}{dt} = \frac{rxy}{1 + \alpha x^2} - d_3 z.$$

is bounded. [4]

6. What is functional response? Draw the curves of Holling type-I, type-II, type-III, and type-IV functional responses. [4]

7. Find the Z-controller $U_{pred}(t)$ of the system

$$\frac{dx}{dt} = rx - d_1 x - px^2 - qxy,$$
$$\frac{dy}{dt} = qxy - my - yU_{pred}(t),$$

so that the prey population $x(t)$ achieves a desired state $x_d(t)$. [4]

8. What is Hopf bifurcation? Prove that the system

$$\begin{aligned} x' &= y, \\ y' &= -x + (\mu - x^2)y, \end{aligned}$$

undergoes Hopf bifurcation by considering μ as the bifurcation parameter. [4]

9. Using center manifold theorem, determine the stability of the origin of the system

$$\begin{aligned} x' &= x^2y - x^3, \\ y' &= -y + x^2. \end{aligned}$$

[4]

10. Show that the system

$$\begin{aligned} x' &= x(x - K)(\alpha - x) - \frac{xy}{1+x}, \\ y' &= -\theta y + \frac{xy}{1+x}, \end{aligned}$$

exhibits bistability behavior. [4]

11. Find the bifurcation points of the system

$$\begin{aligned} x' &= rx - dx - ax^2 - pxy, \\ y' &= -my + pxy, \end{aligned}$$

by taking r as the bifurcation parameter. [4]

12. By constructing a Lyapunov function $v = ax^2 + by^2$ with suitable a and b, show that the system

$$\begin{aligned} x' &= y - x^3, \\ y' &= -x - y^3, \end{aligned}$$

is globally asymptotically stable. [4]

13. Find the interior equilibrium point and Jacobian matrix of the delay model

$$\begin{aligned} \frac{dx}{dt} &= rx(t) - d_1x(t)x(t-\tau) - px(t)y(t), \\ \frac{dy}{dt} &= c_1px(t-\tau)y(t-\tau) - qy(t)z(t) - d_2y(t), \\ \frac{dz}{dt} &= c_2qy(t)z(t) - d_3z(t), \end{aligned}$$

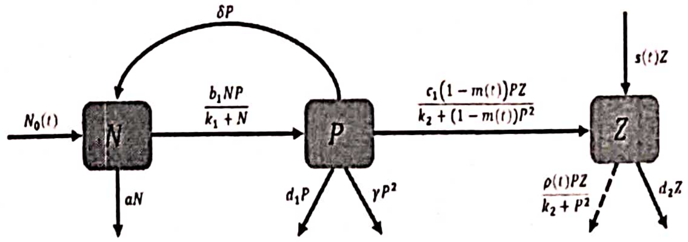
where τ is the delay parameter. [4]

14. Write a MATLAB code to draw the phase portrait and time-series solutions in a single figure for the system

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - \frac{ax^2y}{x^2+b}, \\ \frac{dy}{dt} &= c_1\frac{ax^2y}{x^2+b} - my. \end{aligned}$$

[4]

15. Write down the system of differential equations from the following schematic diagram



[4]