M. A./M. Sc. Examination-2022
Semester-IV
Mathematics
Course: MMC-41(New)
(Differential Geometry and Manifold Theory)

Time: Three Hours | Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. Define a manifold, its dimension, and the first and the second fundamental forms on a manifold. Derive expressions for the coefficients of the first and second fundamentals forms at any point $\mathbf{r}(u^1, u^2)$ in terms of $\mathbf{r}$ and its derivatives. Show that the determinant of a matrix obtained using elements in the second fundamental form can be expressed in terms of metric tensors and their derivatives. [1+1+4+4]

2. Define geodesic curvature $\kappa_g(s)$ of a curve $\mathbf{r}(s)$ $(\equiv \mathbf{r}(u^1(s), u^2(s))$ on the surface $\mathfrak{M}$. State the geometric condition that may provide the equation of geodesic on the surface. Derive expression for $\kappa_g(s)$ in terms of $\mathbf{r}'(s)$, $\mathbf{r}''(s)$ and $\mathbf{U}(s)$. Hence or otherwise find the geodesic on a manifold whose first fundamental form is given by $ds^2 = (du^1)^2 + (du^2)^2$. [1+1+4+4]

3. Define normal curvature and introduce principal curvatures at a point on a manifold, $\mathfrak{M}$ (a smooth surface in $\mathbb{R}^3$). Prove that the principal curvatures of the manifold $\mathfrak{M}$ can be obtained as the eigenvalues of a generalized matrix eigenvalue problem involving matrices with elements as the coefficients present in the first and the second fundamental forms. Hence, find the principal curvatures of the surface $\vec{r}(s,t) = (\rho \sin s \cos t, \rho \sin s \sin t, \rho \cos s), (\rho = \text{const})$ at the point $(s = \frac{\pi}{2}, t = 0)$. [2+5+3]

4. Define the space Λ^m and the exterior derivative(d) of elements in Λ^m. Prove that a) $d : \Lambda^m \to \Lambda^{m+1}$, b) $d^2 \equiv 0$, c) for $\alpha \in \Lambda^p$, $\beta \in \Lambda^q$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$. Establish condition b) stated above for $\alpha = P(x^1, x^2, x^3)dx^1 + Q(x^1, x^2, x^3)dx^2 + R(x^1, x^2, x^3)dx^3$, $P, Q, R \in C^2(\mathbb{R}^3)$. [1+1+2+3+3]

5. Introduce Hodge $*$ map. Derive grad, div, curl, and Laplacian operators combining d-operator and $*$-map. Hence, derive the expression for the Laplacian operator in cylindrical polar coordinates. [1+1+2+3+3]

6. Define the Lie group and Lie group of transformations. Introduce an infinitesimal generator for a one-parameter Lie group of transformations. Define invariant function and the canonical variables for a Lie group of transformations. State the correspondence rules between the one-parameter Lie group of transformations and its infinitesimal generator. Verify your result for the infinitesimal generator $\hat{X} = x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$. [1+1+2+2+4]

M. A./M. Sc. Examination-2022

Semester-IV

Mathematics

Course: MMC-41(Old)

(Differential Geometry and Manifold Theory)

Time: Three Hours | Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. Define normal section of a smooth surface $\mathfrak{S}$ at a point $P(x^1(u^1,u^2), x^2(u^1,u^2), x^3(u^1,u^2))$ on the surface in the direction of a curve $\Gamma(t)$ $(x^1(u^1(t),u^2(t)), x^2(u^1(t),u^2(t)), x^3(u^1(t),u^2(t)))$ on it. Hence find the normal section to the surface $\mathfrak{S} = \{(x^1,x^2,x^3) : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$ at the point $Q(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ along a curve whose tangent is directed along $\mathrm{w}_1 = (1,0,0)$ at Q. Find curvature and torsion to this curve. [2+4+4]

2. Define the first and the second fundamental forms of a smooth surface $\mathfrak{S}$. Find the first and the second fundamental forms for the surfaces $\mathfrak{S}_1 = \{(\sin u^1 \cos u^2, \sin u^1 \sin u^2, \cos u^1)\}$. [5+5]

3. Explain how the Christoffel symbols of the second kind are involved in the second derivative of the position vector of a point on a curve on the smooth surface with respect to the parameters involved. Find the expression for components of these symbols for the surface $\mathfrak{S} = \{(\cos v, \sin v, z) : v \in [0, 2\pi], z \in \mathbb{R}\}$. [2+3+5]

4. Define normal curvature in the direction v at a point Q on a surface $\mathfrak{S}$. Introduce principal curvatures and principal directions. Prove that principal directions on a surface are orthogonal. [1+4+5]

5. Prove the formula $K = \frac{L}{g}$. Find the parameter dependence of Gauss curvature at a point (u,v) on the surface $\mathfrak{S} = (u, v, f(u,v))$. [5+5]

6. Derive the equation for geodesic on a surface, in general. Hence find geodesics on a cylindrical surface joining $(\rho,0,1)$, $(\rho,0,5)$ and $(\rho,0,1)$, $(\rho,\pi,1)$ ($\rho =$ const.). Interpret the nature of (geodesic) curves. [5+4+1]

M. Sc. Examination-2022
Semester-IV
Mathematics
Paper: MMC 42
(Operations Research)

Time: 3 Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) When a linear programming problem (LPP) has no optimal solution? Write down your observation in connection with the revised simplex method (RSM). [2]

(b) Solve the following LPP by revised simplex method (RSM):

$$\begin{aligned} \text{Minimize } f(x_1, x_2) &= 3x_1 + x_2 \\ \text{subject to } 2x_1 + 3x_2 &\geq 2, \\ x_1 + x_2 &\geq 1, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Is the solution non-degenerate basic feasible? [7+1]

2. (a) Comment on the relevant areas of sensitivity analysis in a real-life platform. [2]

(b) Consider the following LPP:

$$\begin{aligned} \text{Maximize } f(x_1, x_2, x_3) &= 4x_1 + 6x_2 + 2x_3 \\ \text{subject to } x_1 + x_2 + x_3 &\leq 3, \\ x_1 + 4x_2 + 7x_3 &\leq 9, \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

(a) Find the range on the value of non-basic variable coefficient c_3 such that the current solution will still remain optimal.
(b) What happens if c_3 is increased to 12? What is the new optimal solution?
(c) Find the effect of changing the objective function to $f(x_1, x_2, x_3) = 2x_1 + 8x_2 + 4x_3$ on the current optimal solution. Is the new optimal solution unique? [2+3+(2+1)]

3. State the special types of integer linear programming problems (ILPPs). In addition, clarify the areas of their applications briefly.
Find the optimum integer solution of the following LPP by Gomory's cutting plane method:

$$\begin{aligned} \text{Maximize } f(x_1, x_2) &= x_1 + 5x_2 \\ \text{subject to } 3x_1 + 4x_2 &\leq 6, \\ x_1 + 3x_2 &\geq 3, \\ x_1, x_2 &\geq 0 \text{ and are integers.} \end{aligned}$$

[2+8]

4. (a) Use the Kuhn-Tucker necessary and sufficient conditions to solve the following nonlinear programming problem (NLPP):

$$\begin{aligned} \text{Maximize } f(x_1, x_2) &= 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2 \\ \text{subject to } x_1 + 2x_2 &\leq 12, \\ x_1 - x_2 &\geq 4, \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

[5]

(b) Solve the following NLPP graphically:

$$\begin{aligned} \text{Maximize } f(x_1, x_2) &= x_1 - (x_2 - 4)^2 + 4 \\ \text{subject to } x_1^2 + x_2^2 &\leq 9, \\ x_1,\, x_2 &\geq 0. \end{aligned}$$

[5]

5. (a) Obtain mathematical expressions for the optimal lot size and corresponding annual cost in an Economic Order Quantity (EOQ) manufacturing model without shortages. [6]

(b) A company manufactures a low-cost speedometer for a motorbike used in its main product line. The demand for the speedometer is 60,000 units per month, and the production rate of the speedometer is 1,50,000 units per month. The carrying cost of the speedometer is Rs. 80 per speedometer per year, and the set-up cost is Rs. 200 per set-up. Find the EOQ and the cycle time. [4]

6. (a) For the queuing model $M/M/1 : \infty/FIFO$, having Poisson input, exponential service, single service channel, infinite capacity of the system, first in first out queue discipline; find the average number of customers in the system at any time. [6]

(b) A one-person barbershop has five chairs to accommodate people waiting for a haircut. Assume customers who arrive when all five chairs are full leave without entering the barbershop. Customers arrive at the average rate of 3 per hour and spend an average of 15 minutes in the barbershop. Then find the probability of a customer getting directly into the barber chair upon arrival. [4]

Use separate answer script for each unit

M. A./M. Sc. Examination-2022

Semester-IV

Mathematics

Elective Course: MME-41 (Applied Stream)

(Electromagetic Theory and Programming in Matlab)

Time: Three Hours Full Marks: 40

Questions are of values as indicated in the margin.

Notations and symbols have their usual meanings.

Unit-I (Electromagnetic Theory)

Answer *any two* questions.

1. (a) Define electric field and potential at a point for a volume charge distribution. Hence derive the Gauss' law and the Poisson equation. [2+4]

   (b) Use the Gauss' law in electrostatics to calculate the electric field, potential and charge density at a point due to the following charge distribution within a sphere of radius r:

$$Q(r) = \frac{qr^2}{a^2}\left(e^{-r/a} - e^{-2r/a}\right),$$

   where a is some constant. Discuss the cases with $r \to 0$ and $r \to \infty$. [4]

2. (a) Calculate the electrostatic potential at a height z above the center of a circular disk of radius a carrying a charge q uniformly distributed over its top surface. [4]

   (b) Define an electric dipole. Hence calculate the electrostatic potential and field at a point due to the dipole charge configuration. [1+5]

3. (a) Show that for a continuous charge distribution in an electric field $\mathbf{E}$, the electrostatic energy density is given by $\mathcal{E} = \frac{1}{2}\varepsilon_0|\mathbf{E}|^2$. [2+1+2]

   (b) Derive the equation of continuity for a volume charge distribution. Define the Lorentz force acting on a charged particle of charge q and mass m. What is Ohm's law? [2+1+2]

Unit-II (Programming in Matlab)

Answer the question no. 1 and *any one* from the rest.

1. Choose the correct alternative (*any five*).

   (a) Index of an array in MATLAB starts with (i) 0 (ii) 1 (iii) Depends on the class of array (iv) Unknown. [2]

   (b) Which MATLAB command is used to clear the command window? (i) clear (ii) clc (iii) close all (iv) clear all. [2]

   (c) – is used to check the equality of two elements in MATLAB. (i) != (ii) == (iii) isequal (iv) =. [2]

   (d) What is the output of $C = A.*B$ when $A = [1\ 0\ 2]$; $B = [3\ 0\ 7]$? (i) [2 0 21] (ii) [3 0 14] (iii) [14 0 3] (iv) [7 0 3]. [2]

   (e) Keys combination used to stop execution of a command in MATLAB. (i) ctrl+c (ii) ctrl+s (iii) ctrl+b (iv) ctrl+enter. [2]

   (f) Which class is used in MATLAB to store a complex number? (i) Double (ii) symbolic (iii) character (iv) array. [2]

   (g) Which one is an assignment operator in MATLAB? (i) '+' (ii) '=' (iii) '*' (iv) '$'. [2]

(h) Which MATLAB command is used to clear all data and variables stored in memory? (i) clear (ii) clc (iii) delete (iv) close. [2]

2. (a) If $A = [1\ 2\ 3; 4\ 5\ 6; 7\ 8\ 9]$ is a matrix, then write the outputs of (i) $A(3,2)$, (ii) $A(2:3,3)$, (iii) $A(2,:)$, and (iv) $A(:,3)$. [4]

   (b) Write the Matlab command to find the roots of a polynomial equation $5x^3 - x + 3 = 0$. [2]

   (c) Write MATLAB codes to find the sum $y_i = x_i + x_{i+1}$, $i = 1, 2, ..., N-1$, where x is the input vector of length N and y is the output vector of pairwise sums. [4]

3. Write a MATLAB program to solve the coupled nonlinear equations: $\dot{x} = ax - bxy$, $\dot{y} = -cy + dxy$, where a, b, c and d are parameters. [10]

Use separate answer script for each unit

M. Sc. Examination-2022

Semester-IV
Mathematics
Elective Course: MME-41 (Pure Stream)
(Galois Theory II and Algebraic Topology)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Marks: 20)
(Galois Theory II)
Answer *any two* questions.

1. (a) Let K be a field. Show that every nonconstant polynomial over K has a splitting field over K. Find a splitting field of the polynomial x^2+x+1 over $\mathbb{Z}_5$ [4+3]

   (b) Find an example of a inseparable polynomial over a finite field. Justify your answer. [3]

2. (a) Let F be a field with p^n elements. Show that $G(F/\mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_n$. [4]

   (b) Let F/K be a finite extension. If F is a splitting field of a separable polynomial then show that $F_{G(F/K)} = K$. Hence show that $F_{G(\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q})} = \mathbb{Q}$. [3+3]

3. (a) Find the Galois group of x^3-2. [7]

   (b) Let K be a field and n be a positive integer. If characteristic of K does not divides n then show that there exists a finite field extension F/K such that F contains a primitive n-th roots of unity. [3]

Unit-II (Marks: 20)
(Algebraic Topology)
Answer *any two* questions.

4. (a) When are two paths α and β in a topological space called path homotopic? Show that the relation of 'being path homotopic' in the set of all paths beginning at x_1 and ending at x_2, where $x_1, x_2 \in X$, is an equivalence relation. [5]

   (b) What do you mean by a contractible space? Explain with examples. Show that a contractible space is necessarily path connected. [5]

5. (a) Define the fundamental group $\pi_1(X,x)$ of a topological space X relative to a base point $x \in X$. Consider the subspace $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ of $\mathbb{R}^3$. Compute $\pi_1(S,(0,0,1))$. [5]

   (b) Prove or disprove: The fundamental groups of a path connected space relative to two distinct base points are isomorphic. [5]

6. (a) Explain a covering space of a given topological space. Deduce that the real line $\mathbb{R}$ is a covering space of the circle $x^2+y^2=1$. [5]

   (b) If X is a covering space of a topological space Y and $c : X \longrightarrow Y$ is the corresponding covering map such that $c(x) = y$, where $x \in X$ and $y \in Y$. Prove that any path in Y beginning at y can be uniquely lifted (i.e. has a unique lifting) to a path in X beginning at x. [5]

M. A./M. Sc. Examination 2022

Semester-IV

Mathematics

Optional Course: MMO-41 (A07) (New Syllabus)

(Magnetohydrodynamics-II)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. For the description of MHD waves in a conducting fluid discuss the notions of characteristic length and time scales, and characteristic speed. [3+4+3]

2. Define plasma beta (β). Discuss the dynamics of MHD waves in three different cases: (i) $\beta \ll 1$, (ii) $\beta \gg 1$, and (iii) $\beta \sim 1$. Hence define two key parameters for the description of MHD waves in plasmas. [1+5+4]

3. Starting from a set of momentum balance equations for two-fluid magnetoplasmas, derive an expression for the generalized Ohm's law. Hence obtain its reduced form in the case of magnetospheric plasmas. [9+1]

4. Use a set of fluid equations to obtain the energy conservation law for MHD waves in a two-fluid plasma. [10]

5. Using an expression for the generalized Ohm's law and the Maxwell's equations obtain an evolution equation for the magnetic field intensity B in a two-fluid plasma. Hence define the magnetic viscosity and the magnetic Reynolds number. Compare these parameters with the ordinary hydrodynamic viscosity and the hydrodynamic Reynolds number. [3+3+4]

6. Write short notes on
   (a) Harmonic waves. [4]
   (b) Polarization of electromagnetic fields. [6]

M. Sc. Examination-2022
Semester-III
Mathematics
Course: MMO-41(A04)
Difference Equations in Ecology

Time: Three Hours Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any ten*** questions.

1. Graphically show the evolution of the solutions of the system

$$x_{t+1} = ax_t + b,$$

where $b \neq 0$ and $|a| = 1$ in $x_t - x_{t+1}$ plane. [4]

2. Solve the difference equation $V_{n+2} - 5V_{n+1} + 6V_n = 7^n + n$. [4]

3. Find the biologically feasible fixed points of the system

$$\begin{aligned} x_{n+1} &= x_n \exp\left(Rm + \frac{R(1-m)}{1+ky_n} - Px_n - D_1 - \frac{A(1-m)y_n}{B+(1-m)x_n}\right), \\ y_{n+1} &= y_n \exp\left(\frac{cA(1-m)x_n}{B+(1-m)x_n} - D_2\right). \end{aligned}$$

[4]

4. Draw the stability triangle in trace-determinant plane for a two-dimensional discrete system. [4]

5. Show that the system

$$\begin{aligned} u_{n+1} &= u_n + u_n(2-u_n)(u_n-1) - pu_nv_n, \\ v_{n+1} &= v_n + pu_nv_n - qv_n. \end{aligned}$$

shows bistability behavior and draw the bistability region in $p-q$ parametric plane. [4]

6. Discuss the stability of the interior fixed point of the system

$$\begin{aligned} u_{n+1} &= u_n + \frac{au_n}{1+kv_n} - du_n - bu_n{}^2 - cu_nv_n, \\ v_{n+1} &= v_n + \mu cu_nv_n - mv_n. \end{aligned}$$

[4]

7. What is Neimark-Sacker bifurcation? Find the Neimark-Sacker bifurcation curve of the system

$$\begin{aligned} x_{n+1} &= x_n exp\left[\frac{r}{1+v} - d - \frac{ax_n}{1+v} - \frac{py_n}{k+v}\right], \\ y_{n+1} &= y_n exp\left[\frac{\mu px_n}{k+v} - m\right]. \end{aligned}$$

[4]

8. Find the flip bifurcation curve of the system

$$\begin{aligned} x_{n+1} &= x_n exp\left[r(1-v) - d - ax_n - \frac{py_n}{k+v}\right], \\ y_{n+1} &= y_n exp\left[\frac{\mu px_n}{k+v} - m\right]. \end{aligned}$$

[4]

9. Solve the two-dimensional discrete-time model with population migration

$$\begin{aligned} x_{n+1} &= 5x_n - 3x_n + 2y_n, \\ y_{n+1} &= 5y_n + 3x_n - 2y_n. \end{aligned}$$

[4]

10. What are the advantages of a dimensionless model? Using suitable transformation, transform the system

$$\begin{aligned} N_{t+1} &= N_t + rN_t\left(1 - \frac{N_t}{K}\right) - cP_tN_t, \\ P_{t+1} &= bP_tN_t, \end{aligned}$$

into the system

$$\begin{aligned} x_{t+1} &= (1+r)x_t - rx_t^2 - ax_ty_t, \\ y_{t+1} &= ax_ty_t. \end{aligned}$$

[4]

11. Using center manifold theorem, determine the stability of the origin of the system

$$\begin{aligned} x_{n+1} &= x_n^2, \\ y_{n+1} &= y_n + x_ny_n. \end{aligned}$$

[4]

12. Find the Z-controller U_n of the system

$$\begin{aligned} x_{n+1} &= x_n exp\left[r - d - ax_n - \frac{py_nx_n}{q + x_n^2}\right], \\ y_{n+1} &= y_n exp\left[\frac{px_n^2}{q + x_n^2} - m\right] - U_ny_n, \end{aligned}$$

so that the prey population x_n achieves a desired state x_d. [4]

13. Suppose there are 100 females in each of the three age classes. Suppose survival rates in years 1 and 2 are 50% and 60%, respectively. Each female in its third year produces 0.8 female offspring. After two years, calculate the population numbers in each group using Leslie matrix. [4]

14. Write MATLAB code to draw the phase portrait and bifurcation diagram (by taking b as a bifurcation parameter) for the system

$$\begin{aligned} x_{n+1} &= x_n + ax_n - bx_ny_n, \\ y_{n+1} &= y_n + \mu_1 bx_ny_n - cy_nz_n - m_1y_n, \\ z_{n+1} &= z_n + \mu_2 cy_nz_n - m_2z_n. \end{aligned}$$

[4]

M. Sc. Examination-2022
Semester-IV
Mathematics
MMO 41 : Optional Paper
Rings and Modules-II

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Every ring R contains unity and every module is unitary.

Answer *any four* questions.

1. (a) Let $J(R)$ be the intersection of all maximal left ideals of R. Show that $J(R) = \{x \in R \mid 1 - rx$ is left invertible for every $r \in R\}$ [3]
   (b) Show that $J(R)$ is an ideal of R and $J(R/J(R)) = (0)$. [4]
   (c) Show that every nil ideal of R is contained in $J(R)$. Hence or otherwise show that $P(R) \subseteq J(R)$. [3]
2. (a) State and prove two equivalent conditions for an ideal to be prime. [3]
   (b) Let $S \subseteq R$ be an m-system. If P is a maximal ideal such that $S \cap P = \emptyset$, show that P is a prime ideal. [4]
   (c) Show that an ideal Q of R is semiprime if and only if it is an intersection of prime ideals. [3]
3. (a) Prove that a ring R is regular if and only if every principal left ideal of R is a direct summand of R. [4]
   (b) Show that a regular ring is semisimple if and only if it is J-semisimple. [3]
   (c) If R is a right artinian ring, then show that $P(R) = J(R)$. [3]
4. (a) State and prove a necessary and sufficient condition for a ring R to be a subdirect product of a family of rings $\{R_\alpha \mid \alpha \in \Delta\}$. [4]
   (b) Show that a commutative ring is primitive if and only if it is a field. Give an example of a noncommutative primitive ring (you need not to justify). [3]
   (c) Show that a ring is semiprime if and only if it is a subdirect product of a family of prime rings. [3]
5. (a) Show that every projective module is direct summand of a free module. [3]
   (b) Prove that a R-module J is injective if and only if for every left ideal L of R, any R-morphism $L \longrightarrow J$ can be extended to an R-morphism $R \longrightarrow J$. [4]
   (c) Let R be a ring. Show that every R-module is projective if and only if every R-module is injective. [3]
6. (a) Give an example of a nil ideal which is not nilpotent. [3]
   (b) Find $J(\mathbb{Z}_{100})$. [2]
   (c) Let $f : R \longrightarrow S$ be a ring homomorphism. If P is a prime ideal of S, then show that $f^{-1}(P)$ is a prime ideal of R. [3]
   (d) Find $\sqrt{(p^n)}$ in $\mathbb{Z}$, where p is a prime integer. [2]

M. Sc. Examination 2022
Semester-IV
Mathematics
Core Course: MMO-41 (A03)
(Computational Fluid Dynamics-II)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Attempt *any four* questions.

1. Find the Blasius solution of the steady boundary layer flow on a semi-infinite flat plate and calculate the coefficient of skin friction on the plate wall. [10]

2. (a) Show that $\int_0^\delta \frac{u}{U} dy = \delta - \delta_1$, (ii)$\int_0^\delta \left(\frac{u}{U}\right)^2 dy = \delta - \delta_1 - \delta_2$, (iii)$\int_0^\delta \left(\frac{u}{U}\right)^3 dy = \delta - \delta_1 - \delta_3$ for a boundary layer flow with free stream velocity U. [3]

   (b) For laminar boundary layer flow past a two-dimensional wedge, the velocity at outer edge of the boundary layer is given by $u_e = Cx^{\beta/(2-\beta)}$, where β is the wedge angle. By introducing the similarity variable $\eta = y\sqrt{u_e/((2-\beta)x\nu)}$ and the dependent variable $f = \psi/\sqrt{((2-\beta)u_e x\nu)}$, where ψ is the stream function, show that the governing equations reduce to $\frac{\partial^3 f}{\partial \eta^3} + f\frac{\partial^2 f}{\partial \eta^2} + \beta\left[1 - \left(\frac{\partial f}{\partial \eta}\right)^2\right] = 0$. [7]

3. Describe Lax-Wendroff's technique using the calculation of density for the solution of unsteady two dimensional Euler equation. [10]

4. Discuss MAC (Marker and Cell) method for primitive variables approach using staggered grid. Also, derive the discretized Poisson equation for pressure field in this formulation. [10]

5. Find the sixth order compact difference scheme for the Laplace equation. [10]

6. Find the compact high-order formulae for the wall boundary conditions at two walls (one horizontal and one vertical) in the case of flow in a rectangular cavity while solving Navier Stokes equations using stream-function vorticity formulation for viscous, incompressible flow. [10]

M. A./M. Sc. Examination-2022
Semester-IV
Mathematics
Course: MMO-41(P02)
(Advanced Functional Analysis)

Time: Three Hours | Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any __four__ questions

1. (a) When is an orthonormal set of vectors in a Hilbert space said to be complete? If $\{e_i\}$ is a complete orthonormal sequence in a Hilbert space H then prove that for each $x \in H$,

    (i) $x = \sum_{i=1}^{n} < x, e_i > e_i$

    (ii) $||x||^2 = \sum_{i=1}^{n} | < x, e_i > |^2$.

    [5]

   (b) Show that every Hilbert space is reflexive. [5]

2. (a) Define sesquilinear form in a linear space.
   Show that inner product in a Hilbert space is a bounded sesquilinear form. [4]

   (b) State and prove Reisz representation theorem for a sesquilinear form in a Hilbert space. [6]

3. (a) Define projection operator over a Hilbert space H and show that a bounded linear operator $T : H \to H$ is a projection operator iff T is self-adjoint and idempotent. [5]

   (b) If $P : H \to H$ is a projection operator then prove that

    (i) $< Px, x >= ||Px||^2, \; x \in H$

    (ii) $||P|| \leq 1$ and $||P|| = 1$ iff $P(H) \neq \{\theta\}$.

    [3]

   (c) Let T be a bounded linear operator defined on a Hilbert space. Show that T is normal iff its real and imaginary parts commute. [2]

4. (a) Define unitary linear operator over a Hilbert space H. Prove that a bounded linear operator T over H is unitary iff T is isometric ans surjective. [4]

(b) If X is a Banach space and $T : X \to X$ is a bounded linear operator with $||T|| < 1$, prove that $(I - T)$ is invertible and

$$(I - T)^{-1} = I + T + T^2 + \dots$$

[4]

(c) Let $T : H \to H$ be a bounded linear operator. Prove that if T is self-adjoint then T is normal but not conversely. [2]

5. (a) Let $T : H \to H$ be a bounded linear self adjoint operator on a complex Hilbert space H. Then show that a number $\lambda \in \rho(T)$ iff $\exists c > 0$ such that for every $x \in H$,

$$||T_\lambda x|| \geq c||x||$$

[6]

(b) Show that spectrum $\alpha(T)$ of a bounded self-adjoint linear operator $T : H \to H$ where H is a complex Hilbert space lies in the closed interval $[m, M]$ where

$$m = \underset{||x||=1}{Inf} < Tx, x >, \; M = \underset{||x||=1}{Sup} < Tx, x >$$

[4]

6. (a) Let $\{e_k\}$ be an orthonormal sequence in an inner product space X. Prove that for every $x \in X$,

$$\sum_{k=1}^{\infty} | < x, e_k > |^2 \leq ||x||^2$$

[5]

(b) Let $\{T_n\}$ be a sequence of compact linear operators defined on a normed linear space into a Banach space Y. If $||T_n - T|| \to 0$ as $n \to \infty$ then show that T is a compact linear operator.

[5]

M. Sc. Examination-2022

Semester-IV

Mathematics

Paper: MMO 41 (A05)

(Dynamics of Ecological System-II)

Time: 3 Hours | Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) What do you mean by Allee effect in an interacting species system?
(b) Comment on the linear stability analysis and influence of Allee effect constant m about the co-existence equilibrium point of the following modified Leslie-Gower predator-prey model:

$$\begin{aligned}\frac{du}{dt} &= r_1u\left(1-\frac{u}{K}-\frac{m}{u+b}\right)-\frac{c_1uv}{u+a_1},\\ \frac{dv}{dt} &= r_2v\left(1-\frac{c_2v}{u+a_2}\right),\\ u(0) &> 0,\ v(0)>0,\end{aligned}$$

where all the system parameters are positive. [2+(4+4)]

2. Explore the influence of discrete time delay in the following predator-prey population dynamics:

$$\begin{aligned}\frac{du}{dt} &= ru\left(1-\frac{u}{K}\right)-\frac{\alpha uv}{\beta+u},\\ \frac{dv}{dt} &= \frac{\gamma u(t-\tau)v}{\beta+u(t-\tau)}-\delta v,\\ u(t=0)=u_0(t) &> 0,\ v(t=0)=v_0(t)>0,\ t\in[-\tau,0],\end{aligned}$$

where $\tau>0$ represents the reaction time of the predation. [10]

3. Reflect on the famous Hastings-Powell three species food chain model:

$$\begin{aligned}\frac{du}{dt} &= ru\left(1-\frac{u}{K}\right)-\frac{c_1a_1vu}{b_1+u},\\ \frac{dv}{dt} &= \frac{a_1uv}{b_1+u}-\frac{a_2vw}{b_2+v}-d_1v,\\ \frac{dw}{dt} &= \frac{c_2a_2vw}{b_2+v}-d_2w,\ 0<c_1^{-1}<1,\ 0<c_2<1,\\ u(0) &> 0,\ v(0)>0,\ w(0)>0,\end{aligned}$$

where u, v and w are the respective population sizes of prey, middle predator and top predator at time t. Write down your remarks around the co-existence equilibrium point on the following issues:
(i) Hopf-bifurcation,
(ii) Chaotic dynamics. [6+4]

4. Obtain the necessary and sufficient conditions of diffusion-driven instability of the following two species interaction:

$$\begin{aligned}\frac{\partial u}{\partial t} &= f_1(u,v)+d_1\nabla^2u,\\ \frac{\partial v}{\partial t} &= f_2(u,v)+d_2\nabla^2v,\\ u(0,x)=u_0(x) &\geq 0,\ v(0,x)=v_0(x)\geq 0,\end{aligned}$$

where $\nabla^2\equiv\frac{\partial^2}{\partial x^2}$ and all the system parameters are positive. [10]

5. Consider the Segel-Jackson type reaction-diffusion system

$$\begin{aligned}\frac{\partial v}{\partial t} &= (1+kv)v - ave + \delta^2\nabla^2 v,\\ \frac{\partial e}{\partial t} &= ev - e^2 + \nabla^2 e,\\ v(0,x) = v_0(x) &\geq 0,\ \ e(0,x) = e_0(x) \geq 0,\end{aligned}$$

where v and e represent concentrations of morphogens at (x,t); $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}$.
(i) Find the non-trivial homogeneous steady state(s).
(ii) Show that the condition for Turing instability is $k - \delta^2 > 2\delta\sqrt{a-k}$. [4+6]

6. Calculate traveling-wave front solutions of the following reaction-diffusion predator-prey system:

$$\begin{aligned}\frac{\partial u}{\partial t} &= ru\left(1-\frac{u}{K}\right) - \frac{auv}{b+u} + d_1\frac{\partial^2 u}{\partial x^2},\\ \frac{\partial v}{\partial t} &= -dv + \frac{cuv}{b+u} + d_2\frac{\partial^2 v}{\partial x^2},\end{aligned}$$

where u representing prey population size at (x,y,t); v representing predator population size at (x,y,t) and all other system parameters are positive. [10]

M.A./M.Sc. Examination-2022
Semester-IV
Mathematics
Optional Course: MMO 41 (A8)
(Mathematical Pharmacology-II)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. Write down the governing equations for a two-compartment model in which a drug is rapidly injected and distributed instantaneously throughout the central compartment and more slowly throughout a peripheral compartment with a first-order elimination kinetics from the central compartment. Obtain the transient concentration in each compartment. [3+7]

2. What is meant by ideal plug flow and ideal stirred tank model? Considering microvascular mixing in the tissue compartment, obtain the expression for drug concentration in the central compartment in an ideal plug flow model. [3+7]

3. Write down the governing equation and appropriate conditions in case of diffusion of a drug through a planar membrane for transdermal delivery. Obtain the unsteady concentration of drug in the polymer and the total amount of drug leaving the membrane. [2+5+3]

4. What is a matrix delivery system? Discuss the pharmacokinetics of T-20 after intravenous administration and obtain the steady state solution for plasma concentration. [5+5]

5. Obtain the concentration inside pores in case of diffusion and chemical reactions inside a porous catalyst. [10]

6. Using appropriate initial and boundary conditions, calculate the total amount of drug (M_t) remaining in an implanted polymer matrix at time t. [10]

M.Sc Examination-2022

Semester-IV

Mathematics

Optional Course: MMO-41(P5)

(Algebraic Coding Theory-II)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer *any four* questions.

1. (a) Let C_i be an $[n_i, k_i, d_i]$-linear code over F_q for $i = 1, 2$. Show that direct sum of C_1 and C_2 defined by $C_1 \cdot C_2 = \{(c_1, c_2) : c_1 \in C_1,\ c_2 \in C_2\}$ is an $[n_1+n_2, k_1+k_2, min\{d_1, d_2\}]$-linear code over F_q. [5]

   (b) Find the generator matrix for the Reed-Muller code $R(1, 2)$. Hence find the generator matrix of $R(1, 3)$. [3+2]

2. (a) Construct a binary $[9, 3, 4]$-linear code. [8]

   (b) Let α be a root of $2 + x + x^2 \in F_3[x]$. Find a minimal polynomial of α^2. [2]

3. (a) Find the corresponding ideal in $F_3[x]/(x^3 - 1)$ of the ternary code $C = \{(0, 0, 0), (1, 1, 1), (2, 2, 2)\}$. [5]

   (b) How many binary cyclic codes of length 6 are there? Find all generating polynomials corresponding to each of such codes. [2+3]

4. (a) Let C be the binary $[7, 4]$-cyclic code generated by $g(x) = 1 + x^2 + x^3$. Find the parity check matrix of C. [5]

   (b) Show that the dimension of a q-ary BCH code of length $q^m - 1$ with designed distance δ generated by $g(x) = \text{lcm}(M^{(a)}(x), M^{(a+1)}(x), \ldots, M^{(a+\delta-2)}(x))$ is independent of the choice of the primitive element a. [5]

5. (a) Give an example of a BCH code of lenth 15 having distance and designed distance 7. [5]

   (b) Find the generator matrix of an 8-array $[7, 5]-$code from its generator polynomial $g(x) = (x - \alpha)(x - \alpha^2)$, where α is a root of $1 + x + x^3 \in F_2[x]$. [5]

6. Show that the generalized RS code $GRS_k(\alpha, v)$ has parameters $[n, k, n - k + 1]$,and hence show that it is an MDS code. [7+3]

M.Sc. Examination 2022
Semester-IV
Mathematics
Paper: MMO 41 (P01)
(Advanced Complex Analysis-II)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Prove that a single valued analytic function which has no other singularities than poles (including point at infinity) is a rational function. [3]
   (b) Let $f(z)$ be meromorphic in $|z| < R$ and $f(0) \neq 0$. If it has zeros at $a_1, a_2, ...a_m$ and poles at $b_1, b_2, ...b_n$ with moduli not exceeding $r(0 < r < R)$ then show that $log\{|\frac{b_1.b_2,.....b_n}{a_1,a_2,.....a_m} f(0)|r^{m-n}\} = \frac{1}{2\pi}\int_0^{2\pi} log|f(re^{i\theta})|d\theta.$ [3]
   (c) State and prove Poisson-Jensen formula. [4]
2. (a) Define Nevanlinna's characteristic function $T(r, f)$. Find $T(r, f)$ when $f(z) = 3e^{2z}$. [3]
   (b) Show that if $0 < |k| < 1$, then $|T(r, f + k) - T(r, f)| \leq log2$. [2]
   (c) State and prove Cartan's identity. [4]
   (d) State Nevanlinna's first fundamental theorem. [1]
3. (a) Prove that for any $'a'$, $N(r, a)$ is a convex function of $logr$. [2]
   (b) If f be a non-constant analytic function, regular in $|z| \leq 2r$, then show that $T(r, f) \leq log^+ M(r, f) \leq 3T(2r, f)$ for all large r. [4]
   (c) If ρ_1 and ρ_2 are the orders of the meromorphic functions f_1 and f_2 respectively, then show that the order of $f_1.f_2 \leq max\{\rho_1, \rho_2\}$ and sign of equality holds if $\rho_1 \neq \rho_2$. [4]
4. (a) If S(r) is of order $k(0 < k < \infty)$ and of convergence class, then show that it is of minimal type. Show by an example the converse is not always true. [3]
   (b) Define the order of a meromorphic function $f(z)$. Find the order of $f(z) = \frac{z+2}{z^2-1}$. [3]
   (c) Let $f(z)$ be a meromorphic function of order ρ. Then prove that for every $'a'$ and arbitrary $\epsilon > 0$
      i) $N(r, a) = O(r^{\rho+\epsilon})$
      ii) $n(r, a) = O(r^{\rho+\epsilon})$ and
      iii) $\sum \left(\frac{1}{r_n(a)}\right)^{\rho+\epsilon}$ is convergent, where $r_1(a), r_2(a), ...$are the moduli of $'a'$-point of $f(z)$ ordered by increasing magnitude. [4]
5. (a) Show that for the function $f(z) = e^{e^z}$, $\frac{T(r,f)}{log^+ M(r,f)} \to 0$ as $r \to \infty$. [3]
   (b) If f is a non-constant meromorphic function in the complex plane and $'a'$ be any finite complex number, then define $\delta(a, f), \theta(a, f)$ and $\Theta(a, f)$. Find also the relation between them. [4]
   (c) If $f(z)$ is a non-constant meromorphic function in the complex plane then show that there exist at most four distinct values $a_i(i = 1, 2, 3, 4)$ such that $f(z) = a_i$ has only multiple roots. [3]
6. (a) If f and g are two non-constant polynomials share '1' CM and $f(0) = g(0) \neq 1$, then show that $f \equiv g$. [3]
   (b) State and prove Nevanlinna's five point uniqueness theorem. [5]
   (c) Define an elliptic function and give an example of it. [2]

M.A./M.Sc. Examination - 2022
Semester - IV
Mathematics
Course: MMO - 41(A-12)
(Theory of Computation - II)

Time: Three Hours
Full Marks: 40

Questions are of value as indicated in the margin
Notations and symbols have their usual meaning

Answer *any four* questions

1. (a) Define a phrase structure grammar G and the language $L(G)$ generated by G. Construct a grammar G generating the language $\{a\}^+$. [2]

   (b) Let a grammar G be defined by the productions
   $$P = \{S \to 0SA2, S \to 012, 2A \to A2, 1A \to 11\}.$$ Find $L(G)$. [2]

   (c) Show that a context-sensitive language is recursive. [3]

   (d) Show that the class $\mathcal{L}_0$, language of type-0 grammar, is closed under union. [3]

2. (a) Define a regular expression. Construct a minimal state finite automaton corresponding to the regular expression $(a \mid b)^*abb$. [1+5]

   (b) Show that the set $\{ww \mid w \in \{0,1\}^*\}$ is not regular. Construct a regular grammar G generating the regular set $0^*1(0 \mid 1)^*$. [2+2]

3. (a) Construct a deterministic finite automaton accepting all strings over $\{0,1\}$ ending in 010. [3]

   (b) Construct a deterministic finite automaton equivalent to the grammar
   $$S \to aS \mid bS \mid aA,\ A \to bB,\ B \to aC,\ C \to \epsilon.$$ [3]

   (c) Define a context-free grammar. When is a context-free grammar ambiguous? Show that the grammar $S \to aB \mid ab,\ A \to aAB \mid a,\ B \to ABb \mid b$ is ambiguous. [4]

4. (a) Construct a grammar in Chomsky normal form generating the language
   $$\{wcw^T \mid w \in \{a,b\}^*\}.$$ [4]

   (b) State Pumping lemma for context-free language. Show that the language
   $$L = \{a^p \mid p \text{ is a prime}\}$$ is not context-free. [1+2]

   (c) Construct a push-down automaton for the language $\{wcw^T \mid w \in \{a,b\}^*\}$. [3]

5. Explain parsing in connection with context free grammar. Construct a top-down parser for the grammar of arithmetic expressions involving $+$ and $*$ generated by the productions
   $$E \to E + T \mid T$$
   $$T \to T * F \mid F$$
   $$F \to (E) \mid id.$$
   Find a left-most derivation of $id + id * id$. [1+7+2]

6. (a) Define primitive and partial recursive functions over N. Show that x^y is primitive recursive. Compute $U_2^3(1,2,1)$. [1+1+2+1]

   (b) Construct a Turing machine which accepts/recognizes the language $\{1^n 2^n 3^n \mid n \geq 1\}$. [5]

M. A./M. Sc. Examination-2022

Semester-IV

Mathematics

Optional Course : MMO-41 (P03)(New Syllabus)

(Advanced Real Analysis-II)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer **question no.6** and any **three** from the rest

1. (a) Suppose μ_1 and μ_2 are measures such that at least one is finite. Show that $\mu_1 - \mu_2$, is a signed measure. [3]

   (b) State and prove Hahn decomposition theorem for signed measure. Show that Hahn decomposition is not unique. [5+1]

   (c) Suppose μ is a signed measure on $(X, \mathcal{A})$. Then show that for every $A \in \mathcal{A}$, $|\mu|(E) = \sup\{\sum|\mu(E_i)| : \{E_i\}$ *is a pairwise disjoint sequence of sets with* $E = \cup E_i\}$ [3]

2. (a) Show that there is an outer measure on $\mathbb{R}^2$, constructed from a pre measure under which open sets in $\mathbb{R}^2$ need not be measurable. [4]

   (b) When is a measure space $(X, \mathcal{M}, \mu)$ said to be complete? Show that there is a complete measure space $(X, \overline{\mathcal{M}}, \overline{\mu})$ such that $\mathcal{M} \subset \overline{\mathcal{M}}$ and $\overline{\mu}$ is extension of μ on $\overline{\mathcal{M}}$. [8]

3. (a) Suppose ν is an additive set function defined on an algebra $\mathcal{A}$. Define upper, lower, and total variations of ν, also show that they are additive set functions. [3+4]

   (b) Show that construction of outer measure using method-II always gives rise to a metric outer measure. [5]

4. (a) Define Lebesgue Stieltjes outer measure μ_f^*. Find $\mu_f^*[a,b)$, $\mu_f^*(a,b)$, $\mu_f^*\{a,b\}$ and $\mu_f^*\{a\}$. [2+5]

   (b) If f is integrable on the sets E and F, $E \cap F = \emptyset$ then show that f is integrable on $E \cup F$ and $\int_{E\cup F} f d\mu = \int_E f d\mu + \int_F f d\mu$ [5]

5. (a) Suppose (A, B) is Hahn decomposition of X, and E is a measurable set. Let $f = \chi_{E\cap A} - \chi_{E\cap B}$. Show that $\int_E f d\mu = |\mu|(E)$. [3]

   (b) Suppose f is defined on $(X, \mathcal{M}, \mu)$ and $\int_X f$ exists. Let $\nu(E) = \int_E f d\mu$. Then show that ν is a signed measure on X. Also find Hahn and Jordon decomposition of ν. [5+4]

6. Answer any two

   (a) Show that Lebesgue outer measure on $\mathbb{R}$ is regular. [2]

   (b) Show that a completely additive set function ψ on an algebra is non decreasing if and only if ψ is non negative. [2]

   (c) Let f be a measurable function defined on $(X, \mathcal{M}, \mu)$. Define $\int_E f d\mu$ for any set $E \subset X$. [2]

M. A./M. Sc. Examination-2022
Semester-IV
Mathematics
Optional Paper: MM0-41(P11/A13)
(Weak Formulation of Elliptic Partial Differential Equation)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. Define test function and distribution. Verify which among the functions a) $\phi_1(x) = e^{-x^2}$, b) $\phi_2(x) = \begin{cases} 0 & x > 1 \\ e^{-\frac{1}{1-x^2}} & x < 1 \end{cases}$ c) $\phi_3(x) = e^{-|x|}$ may be regarded as a test function or a distribution. Determine their natures (regular or singular), whenever distribution. Define the order of a distribution. What will be the order of a regular distribution? [1+1+3+3+1+1]

2. Define Heavyside function, $H(x)$. Express $H(\tanh(-x))$ in terms of $H(x)$. Prove that derivative of distribution follows derivation law. Express $(f(x)H(x))'''$ in terms of f at 0, $H(x), \delta(x)$ and their derivatives. [1+2+2+5]

3. a) Verify the result $\int_{-2\pi}^{2\pi} e^{\pi x}\delta(x^2 - \pi^2)dx = \frac{1}{\pi}\cosh\pi^2$,
b) Prove that for any classical function $g(x)$, $g(x) + a_0\delta(x-\xi) + a_1\delta'(x-\xi) + \cdots + a_5\delta'''''(x-\xi) = 0$ on $\mathbb{R}$ implies $g(x) = 0, a_0 = a_1 = \cdots = a_5 = 0$. [4+6]

4. Define tempered distribution. Mention similarities and differences between tempered distribution and the distribution mentioned earlier, if exists with some illustration. Prove that Fourier transform of any $\phi \in \mathcal{S}$ (space of function with rapid decay) is also an element in $\mathcal{S}$. Establish the formula $\widehat{\frac{d^k t}{dx^k}}(u) = (-iu)^k\hat{t}(u)$ for any tempered distribution $t \in \mathcal{S}'$. [1+2+3+4]

5. Define convolution of two distributions. Prove the formula $(D^k s) * t = D^k(s * t) = s * (D^k t)$. Express $f(x) = \int_{-\infty}^{x}\phi(t)dt, \phi \in \mathcal{S}$ as the convolution two distributions. Prove that $t(x) = E * \tau$ may be regarded as the solution of $\frac{d^2t}{dx^2} = \tau(x)$ for $\tau \in \mathcal{D}'$ where E is the fundamental solution to a differential equation to be determined by you. [1+3+2+4]

6. State Lax-Milgram lemma. Exercise this result for determining the norm of the space of weak solution to the boundary value problem $-u''(x) + \lambda\, u(x) = f(x)$ on $\Omega = (a, b) \subset \mathbb{R}$ with $u(a) = 0 = u(b)$. [2+8]