M. Sc. Examination-2022

Semester-I

Mathematics

Paper: MMC-11

(Real Analysis)

Time: 3 Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Define the outer measure $m^*(A)$ of a set $A \subseteq \mathbb{R}$. Prove that m^* is finitely subadditive.
   (b) If $A_1, A_2, \ldots$ are measurable subsets of $\mathbb{R}$, examine if $\bigcup_{n=1}^{\infty} A_n$ is measurable. [(2+4)+4]

2. (a) Describe Lebesgue measure $m(E)$ of a set $E \subseteq \mathbb{R}$. Let $A_1, A_2, \ldots$ be measurable subsets of $\mathbb{R}$ having finite measures such that $A_1 \supseteq A_2 \supseteq \ldots$. Show that $m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$.
   (b) Let $A \subseteq \mathbb{R}$ be a measurable set and f be an extended real valued measurable function on A and $f = g$ *a.e.* on A. Prove that g is measurable on A. [(1+5)+4]

3. (a) Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \longrightarrow f$ pointwise *a.e.* on a set $A \subseteq \mathbb{R}$. Check if f is measurable on A.
   (b) If $f : E \longrightarrow \mathbb{R}$ is measurable on $E \subseteq \mathbb{R}$, then show that for each $\delta > 0$, there exist a continuous function g on $\mathbb{R}$ and a closed set $F \subset E$ such that $f = g$ on F and $m(E \backslash F) < \delta$. [5+5]

4. (a) For each $n \in \mathbb{N}$, let $f_n : A \longrightarrow \mathbb{R}$ be a bounded measurable function on A, where $m(A) < \infty$. If $f_n \longrightarrow f$ uniformly on A, show that $\int_A f_n \longrightarrow \int_A f$. Does the result hold good if $f_n \longrightarrow f$ pointwise only on A? Justify your answer.
   (b) Establish Chebychev's inequality. [(4+2)+4]

5. (a) Let $\{f_n\}$ be a sequence of non-negative measurable functions on a set $E \subseteq \mathbb{R}$ such that $f_n \longrightarrow f$ pointwise *a.e.* on E. Show that $\int_E f \leq \liminf_{n\to\infty} \int_E f_n$. Further, if $\{f_n\}$ is monotone increasing, prove that $\int_E f = \lim_{n\to\infty} \int_E f_n$.
   (b) Let f be an extended real valued function on $\mathbb{R}$ such that $f = \infty$ on $\mathbb{Q}$. Determine the value of $\int_{\mathbb{Q}} f$. [(4+2)+4]

6. (a) Let $m(E) < \infty$ and $\{f_n\}$ be a sequence of measurable functions on E such that $f_n \longrightarrow f$ pointwise *a.e.*. on E and f is finite *a.e.* on E. Examine if $f_n \longrightarrow f$ in measure on E.
   (b) Establish the linearity and monotonicity of integration for non-negative measurable functions over a measurable subset of $\mathbb{R}$. [3+7]

M.Sc. Examination 2022

Semester-I

Mathematics

Paper: MMC 12

(Complex Analysis)

Time: 3 Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Let $f(z)$ be a continuous function in a simply connected domain D and z_0 be any arbitrary but fixed point in D. If $F(z) = \int_{z_0}^{z} f(w)dw$, where the integral is independent of the path so long as the path lies in D, then show that $F'(z) = f(z)$ for every z in D. [3]
   (b) State and prove the maximum modulus theorem. [5]
   (c) Using maximum modulus theorem show that $f(z) = z^{10} + z^5 - 7$ has at least one zero in $|z| < 2$. [2]

2. (a) Using Taylor's theorem expand $f(z) = \frac{1}{z-1}$ in powers of $(z+1)$. [2]
   (b) State and prove Laurent's theorem. [5]
   (c) Expand $f(z) = \frac{1}{(z+2)(z-3)}$ in Laurent series in the region $2 < |z| < 3$. [3]

3. (a) Show that if $f(z) = (z-\beta)^m \phi(z)$, where $\phi(z)$ is analytic at β and $\phi(\beta) \neq 0$ then β is a zero of $f(z)$ of order m and conversely. [4]
   (b) State and prove Riemann's theorem on removable singularity. [3]
   (c) If $f(z)$ has a pole at α of order m then show that $|f(z)| \to \infty$ as $z \to \alpha$ in any manner. [3]

4. (a) Determine the nature of the singularities of $sec\frac{1}{z}$. [3]
   (b) Prove that the zeros of a non-constant analytic function are isolated points. [2]
   (c) State and prove Cauchy's residue theorem. Using this theorem evaluate $\oint_{|z|=1} \frac{e^z}{2z-1} dz$. [3+2]

5. (a) State and prove Rouche's theorem(Principle of Argument may be assumed). [3]
   (b) Show that the equation $e^z = 3z^n$, where n is a positive integer has n roots inside the circle $|z| = 1$. [3]
   (c) State and prove open mapping theorem. [4]

6. (a) Prove that a function which is analytic every where including the point at infinity is a constant. [3]
   (b) Find $Res[f(z), \infty]$, where $f(z) = \frac{1}{z^2-1}$. [2]
   (c) Evaluate any one of the following by the method of contour integration:
   (i) $\int_0^{\infty} \frac{\cos x dx}{x^2+a^2}, a < 0$; (ii) $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}, a > b > 0$. [5]

M. Sc. Examination-2022

Semester-I

Mathematics

MMC-13(Linear Algebra)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.

(V is a finite dimensional vector space over F.)

Answer *any four* questions.

1. (a) Show that similar matrices have the same eigen values. Let λ be an eigen value of two similar matrices A and B. Show that $\dim E_\lambda(A) = \dim E_\lambda(B)$ [2+2]

   (b) Let $A, B \in M_n(F)$. Show that $\chi_{AB}(t) = \chi_{BA}(t)$. Give examples of two matrices A and B such that AB and BA are not similar. [3+1]

   (c) Show that 0 is an eigen value of the matrix $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 2 & -1 & 10 \end{pmatrix}$ and the geometric multiplicity of 0 is 1, without solving the characteristic polynomial and finding the eigen space E_0. [2]

2. (a) Let $w \in \mathbb{R}^n$ be such that $\|w\| = 1$. Find the eigen values and eigen vectors of the Householder matrix $H_w = 1 - 2ww^T$. [4]

   (b) Let $T : V \longrightarrow V$ be a linear transformation, $\alpha \in V$ and $W =< \alpha >_T$. If k is the largest positive integer such that $\{\alpha, T(\alpha), \cdots, T^{k-1}(\alpha)\}$ is linearly independent, then show that $\dim W = k$. [3]

   (c) Let $T : V \longrightarrow V$ be a linear transformation and W be a T-invariant subspace of V. Show that the characteristic polynomial of T_W divides that of T. [3]

3. (a) Let $m(t)$ be the minimal polynomial of the linear operator $T : V \longrightarrow V$. Show that $\lambda \in F$ is an eigen value of T if and only if $m(\lambda) = 0$. [2]

   (b) Let $T : V \longrightarrow V$ be a linear transformation, $m(t)$ be the minimal polynomial of T and $\deg m(t) = k$. Show that dim $span(\{I, T, T^2, \cdots\}) = k$. [3]

   (c) Let $T : V \longrightarrow V$ be a linear transformation, and U and W be two T-invariant subspaces of V such that $V = U \oplus W$. Show that the minimal polynomial of T is the least common multiple of the minimal polynomials of T_U and T_W. [3]

   (d) Show that every reflection is diagonalizable. [2]

4. (a) Let $A \in M_n(F)$. Show that A is diagonalizable if and only if F^n has a basis of eigen vectors of A. [3]

   (b) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$. Find a nonsingular matrix P such that $P^{-1}AP$ is a diagonal matrix. [3]

   (c) Let $T : V \longrightarrow V$ be a linear transformation and $f(t) \in F[t]$. If $f(t) = g(t)h(t)$ in $F[t]$ such that $\gcd(g(t), h(t)) = 1$, then show that $\ker f(T) = \ker g(T) \oplus \ker h(T)$. [4]

5. (a) Show that two projections are similar if and only if they are of same rank. Does this result hold for reflections? [2+1]

   (b) Let $T : V \longrightarrow V$ be a linear operator and $\beta = \{v_1, v_2, \cdots, v_n\}$ be a basis of V. Prove that $[T]_\beta$ is upper triangular if and only if $span(\{v_1, v_2, \cdots, v_k\})$ is T-invariant for all $1 \leqslant k \leqslant n$. [4]

   (c) Give examples of two diagonalizable linear operators S and T such that $S + T$ is diagonalizable. [3]

6. (a) Let $T : V \longrightarrow V$ be a linear operator and $\chi(t) = (t - \lambda_1)^{n_1}(t - \lambda_2)^{n_2} \cdots (t - \lambda_k)^{n_k}$. Define generalized eigen spaces G_{λ_i} of T. Show that λ_i is the only eigen value of T over G_{λ_i}. [3]

   (b) Let N be an $n \times n$ matrix in Jordan canonical form. Show that the number of Jordan blocks in N of order k is $2n(A^k) - n(A^{k-1}) - n(A^{k+1})$ for all $1 < k < n$. [3]

   (c) Let $N = J_3(0) \oplus J_2(0) \ominus J_2(0) \oplus J_1(0)$. Find the index of nilpotence, Segre characteristic, Weyr characteristic and rank of N. Also find the geometric multiplicity of the eigen value 0. [4]

Use separate answer script for each unit

M.Sc. Examination-2023

Semester-I
Mathematics
MMC-14
(Ordinary Differential Equations)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Define a Sturm-Liouville Problem. Under what conditions a Sturm-Liouville problem is called a Regular or a Singular? [1+1]
   (b) Find the eigen values and orthonormal eigen functions of the Sturm-Liouville problem $\frac{d^2y}{dx^2} + \lambda y = 0$ $y(0) = 0, y(\pi) = 0$. [4]
   (c) Draw the phase diagram for the linear harmonic undamped oscillators represented by $\ddot{x} + \omega^2 x = 0$ [4]

2. (a) Solve the given system by constructing Green's function, $\frac{d^2y}{dx^2} - y = -1, 0 < x < 1, y(0) = 0, and y(1) = 0$. [4]
   (b) Convert the given IVP $y'' - y'sin(x) + e^x y = x$ into an integral equation with the initial conditions $y(0) = 1, y'(0) = -1$ [2]
   (c) Under what condition the system $x' = -2tx^2$, on $\mathbb{R}$ has Global or Maximal solution. [2]
   (d) State Peano's Theorem for the existence of first order IVP. Is the condition necessary? Justify your answer with a suitable example. [2]

3. (a) State Picard's Theorem and Lipchitz condition for first order IVP. Which one is stronger? Justify your answer with a suitable example. [2]
   (b) If S is defined by the rectangle $|x| \le a, |y| \le b$. show that the function $f(x, y) = xsin(y) + ycos(x)$, satisfies the Lipschitz condition and find the Lipschitz constant. [3+1]
   (c) Show that $e^{At} = \Phi(t)\Phi^{-1}(0)$, where e^{At} is the Matrix Exponential Function of the system $\dot{\mathbf{x}} = A\mathbf{x}$ and Φ is the Fundamental Matrix of the system. [4]

4. (a) Let f be a solution of $\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + Q(t)x = 0$, having first derivative f' on $a \le t \le b$. Suppose f has an infinite number of zeros on $a \le t \le b$. Then show that $f(t) = 0$ for all t on $a \le t \le b$. [2]
   (b) Apply Picard's Method to solve the following initial value problem upto third approximation of the solution, $\frac{dy}{dx} = 2y - 2x^2 - 3$ given that $y = 2$ when $x = 0$. [4]
   (c) Find the General solution of the following linear homogeneous system using eigen value-eigen vector method.
   $\dot{x} = 6x - 3y$,
   $\dot{y} = 2x + y$. [4]

5. (a) What is a Wronskians ? State Abel's Theorem. Hence show that the Wronskians is either zero everywhere or zero nowhere. [1+1+2]
   (b) Obtain the adjoint equation of the following differential equation. $(1 + x^2)\frac{d^2u}{dx^2} + x\frac{du}{dx} - 4u = 0$. [3]
   (c) Find the nature and stability of the fixed points of
   $\dot{x} = -ax + y$
   $\dot{y} = -x - ay$, for different values of the parameter a. [3]

6. (a) Does $\Phi(t) = \begin{bmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{bmatrix}$ is a Fundamental Matrix for the system $\dot{\mathbf{x}} = A\mathbf{x}$? If so, then find the matrix A. Also find the solution of the system. [3]
   (b) Find the Natural Fundamental set of solutions of the following Differential Equation. $(1 + t^2)y'' - 2ty' + 2y = 0$, with the initial time 1. [3]
   (c) Consider a second-order homogeneous linear differential equation and write its adjoint equation. When a differential equation is called self-adjoint? Find a necessary and sufficient condition that a second-order linear differential equation $a_0(t)\frac{d^2x}{dt^2} + a_1(t)\frac{dx}{dt} + a_2(t)x = 0$ will be self-adjoint. [4]

M. Sc. Examination-2022

Semester-I

Mathematics

Paper: MMC 15

(Partial Differential Equations)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notation and symbol have their usual meanings.

Answer ***any four*** questions.

1. (a) If $u(x,y)$ is the solution of the Cauchy problem $yu_x + xu_y = 0,\ u(0,y) = e^{-y^2}$, then find the value of $u(1,2)$. State the region of xy-plane in which the solution can be determined uniquely.
(b) Solve the following PDE by Monge's method:

$$2x\ u_{xx} - (x+2y)\ u_{xy} + y\ u_{yy} = \frac{x+2y}{x-2y}\ (2u_x - u_y).$$

(c) Find the nature of the PDE $\frac{1}{4}u_{xx} + \sqrt{2022}\ u_{xy} + \{2022 - sgn(y)\}u_{yy} = 0$, for $y > 0$ and $y < 0$.

[(2+1)+5+2]

2. (a) Show that a linear PDE of the type $\sum_{r,s} C_{rs}x^r y^s \frac{\partial^{r+s}u}{\partial x^r \partial y^s} = f(x,y)$ may be reduced to one with constant coefficients by the substitutions $\xi = \ln x,\ \eta = \ln y$.
Hence, find the complete integral of the PDE

$$x^2u_{xx} - xyu_{xy} - 2y^2u_{yy} + xu_x - 2yu_y = \ln\frac{y}{x} - \frac{1}{2023}.$$

(b) Find a surface satisfying $u_{xx} - 2u_{xy} + u_{yy} = 6$ and touching the hyperbolic paraboloid $u = xy$ along its section by the plane $x = y$. [(2+4)+4]

3. (a) Using Cauchy's method of characteristics, find an integral surface of the PDE

$$u_y + uu_x = -x,\ y \geq 0$$
$$u(x,0) = f(x),\ -\infty < x < \infty.$$

(b) Convert the following PDE to the canonical form, and hence solve it:

$$x^2(y-1)\ u_{xx} - x(y^2-1)\ u_{xy} + y(y-1)\ u_{yy} + xy\ u_x - u_y = 0.$$

[5+5]

4. (a) Solve the following homogeneous wave equation by the method of Laplace transform:

$$9\ u_{tt} = u_{xx},$$

$u(0,t) = f(t)$, $lim_{x\to\infty}u(x,t) = 0$, $x,t \geq 0$ where u is the deflection of a string, released from rest on the x-axis $[u(x,0) = 0 = u_t(x,0)]$.

(b) Use the complex form of the Fourier transformation to show that $u = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\xi)e^{-\frac{(x-\xi)^2}{4t}}d\xi$ is the solution of the PDE

$$u_t = u_{xx},\ -\infty < x < \infty,\ t > 0$$

with $u = f(x)$ when $t = 0$ and $u(x,t)$ is bounded. [5+5]

5. (a) State maximum-minimum principle for a function $u(x,y)$ which is continuous in a closed region $\overline{\mathbb{R}}$ (=$\mathbb{R}\cup\partial\mathbb{R}$) and satisfies the Laplace equation $\nabla^2 u = 0$ in the interior of $\mathbb{R}$.
(b) Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$ with boundary $\partial\Omega$. If $u(x,y)$ is the solution of the Dirichlet problem $u_{xx} + u_{yy} = 0$ in Ω and $u(x,y) = 2x^2 - 1$ on $\partial\Omega$. Find the value of $u(\frac{1}{2}, \frac{1}{2})$.
(c) By using the method of separation of variables, obtain the solution of the following initial boundary value problem (IBVP) for the wave equation:

$$u_{tt} = u_{xx},\ x \in (0,\pi), t > 0,$$

BC's: $u(0,t) = u(\pi,t) = 0,\ t > 0,$
IC's: $u(x,0) = \sin x - \sin 2x,\ u_t(x,0) = 0,\ 0 \leq x \leq \pi.$ [2+3+5]

6. (a) If ϕ is a harmonic function in $\mathbb{R}$ and $\frac{\partial \phi}{\partial n} = 0$ on $\partial\mathbb{R}$, then ϕ is a constant in $\overline{\mathbb{R}}$ $(= \mathbb{R} \cup \partial\mathbb{R})$ - prove it.

(b) Solve the following exterior Dirichlet problem for a circle:
PDE: $\nabla^2 u = 0$, $r > a$, $0 \leq \theta < 2\pi$
BC: $u(a, \theta) = f(\theta)$, $0 \leq \theta < 2\pi$
where $f(\theta)$ is a continuous function on $\partial\mathbb{R}$ and $u(r, \theta)$ must be bounded as $r \to \infty$.
Hence deduce Poisson's integral formula of the form
$u(r, \theta) = \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \frac{(r^2 - a^2) f(\phi)}{r^2 - 2r a cos(\phi - \theta) + a^2} \, d\phi$. [3+(5+2)]

M. Sc. Examination-2022

Semester-I

Mathematics

Paper: MMC-16

(Integral Transforms and Integral Equations)

Time: Three Hours Full Marks: 40

Questions are of values as indicated in the margin.

Notations and symbols have their usual meanings.

Answer Question No. 1 and any *three* from the rest.

1. Answer any *five* questions from the following: [5 × 2 = 10]
   (a) Find the inverse Laplace transform of $F(s)$, where $F(s) = \frac{1}{s^2+5s+4} + \frac{e^{-\pi s}}{s^2-1}$. [2]
   (b) Find the inverse Fourier transform of $F(s)$, where $F(s) = \frac{1}{3+4is-s^2}$. [2]
   (c) Find the value of λ for which the integral equation $u(x) = e^{2x} + 3\lambda \int_0^1 e^{2x}e^{2t}u(t)dt$ has infinitely many solutions. [2]
   (d) Solve the Fredholm integro-differential equation $u'(x) = 9x^2 + \int_0^1 u(t)dt,\ u(0) = 1.$ [2]
   (e) Find the Laplace transform of $f(t)$, where $f(t) = \sin t$ when $0 < t < \pi$ and $f(t) = \cos t$ when $t > \pi$. [2]
   (f) Using modified Adomian decomposition method, solve the integral equation $y(x) = 2x + 1 + \sin x + x^2 - \cos x - \int_0^x y(t)dt.$ [2]
   (g) Find the value of $u(1) + u(3)$, where $u(x)$ satisfies the equation $1 + x - e^x = \int_0^x (t - x)u(t)dt.$ [2]

2. (a) Find the resolvent kernel of the integral equation $y(x) = e^x \sin x + \int_0^x \frac{2+\cos x}{2+\cos t} y(t)dt.$ [3]
   (b) Find the Fourier sine transform of $f(x)$, where $f(x) = \cos 9x$ when $0 \leq x \leq 5$, and $f(x) = 0$ when $x > 5$. [2]
   (c) Solve Volterra integral equation of first kind $\frac{x^2}{2} = \int_0^x (1 - x^2 + t^2)\, y(t)dt.$ [2.5]
   (d) Convert the integral equation $y(x) = \sin x - \frac{1}{2}\int_0^x (x-t)^2 y(t)dt$ to an equivalent differential equation. [2.5]

3. (a) State Fredholm alternative. Find the conditions for which the integral equation $y(x) = f(x) + \lambda \int_0^\pi cos(x+t)y(t)dt$ has infinitely many solutions. [1+4]
   (b) Using Laplace transform, solve the differential equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = h(t); y(0) = 0, y'(0) = 0$, where $h(t) = 1, 0 < t < \pi$, and $h(t) = 0, t > \pi$. [5]

4. (a) If f and g are piecewise continuous on $[0,\infty)$ and of exponential order then prove that $L\{f*g\}=L\{f\}.L\{g\}$.
Using the above result prove that $\beta(m,n)=\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$ where $m,n>0$. [2+1]

(b) Evaluate: $\int\limits_0^\infty te^{-3t}\sin 7t\,dt$. [2]

(c) Using Hilbert-Schmidt theorem, solve the integral equation $y(x)=x^2+x+1+\frac{3}{2}\int\limits_{-1}^{1}(xt+x^2t^2)y(t)dt$. [5]

5. Solve the Volterra integral equation $y(x)=x^2+\int\limits_0^x(x-t)y(t)dt$ by

(a) Laplace transform method. [2]
(b) series solution method. [2]
(c) Adomian decomposition method. [2]
(d) successive approximations method. [2]
(e) successive substitutions method. [2]

6. (a) Using Fourier transform solve the differential equation $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+5y=\delta(t)$. [3]

(b) Find the Fourier transform of $f(x)$, where $f(x)=1-x^2$, $-1<x<1$, and $f(x)=0$, otherwise.
Hence using Persaval's identity, evaluate $\int\limits_0^\infty\left(\frac{\sin x-x\cos x}{x^3}\right)^2dx$. [3+2]

(c) Find the Laplace transform of the function $f(t)=t^{5/2}+\sin t\ \cos t+t\sin h5t+e^{-5t}\cos 3t$. [2]