M. Sc. Examination-2023

Semester-II

Mathematics

Course: MMC-21

(Functional Analysis)

Time: 3 Hours

Full Marks: 40

Questions are of value as indicated in the margin
Notations and symbols have their usual meanings

Answer any <u>four</u> questions

1. (a) If (X, d_1) and (Y, d_2) are two metric spaces where (X, d_1) is compact and $f : (X, d_1) \to (Y, d_2)$ is a continuous function, show that image $f(X)$ is a compact subset of Y. Hence deduce that every real-valued continuous function over a bounded closed interval is bounded and attains its bounds. [5]

   (b) Let A be a compact subset in a metric space (X, d). Show that for every subset B of X there exists a point $x_0 \in A$ such that $d(x_0, B) = d(A, B)$. [3]

   (c) If a discrete metric space (X, d) is compact then show that X contains finitely many points. [2]

2. (a) Prove that a subset M in a finite dimensional normed linear space is compact iff M is closed and bounded. [4]

   (b) Show that l_2 is a Banach space with respect to a norm to be defined by you. [4]

   (c) Show that $\mathbb{R}^2$ is a normed linear space with respect to $||\cdot||$ defined by

$$||x|| = \max\{\frac{|x_1|}{a}, \frac{|x_2|}{b}\}$$

   where $a, b > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$. [2]

3. (a) When is a linear operator T over a normed linear space said to be bounded? Prove that every linear operator defined on a finite dimensional normed linear space is bounded. [4]

   (b) If T is a bounded linear operator defined on a normed linear space X to another normed linear space Y, then show that

$$||T|| = \sup_{x \in X, ||x||=1} ||Tx||.$$

   [3]

   (c) If $T_1, T_2 : X \to Y$ are bounded linear operators where X and Y are normed linear spaces then prove that

   (i) $||T_1 + T_2|| \leq ||T_1|| + ||T_2||$

   (ii) $||T_1 \; T_2|| \leq ||T_1|| \; ||T_2||$.

   [3]

4. (a) State and prove Uniform boundedness principle theorem. [7]

   (b) Let $T : C[0,1] \to \mathbb{R}$ defined by $T(x) = \int_0^1 x\, dt,\ x \in C[0,1]$. Show that T is a bounded linear operator. [3]

5. (a) State and prove Bessel's inequality in a Hilbert space. [4]

   (b) Show that $\mathbb{C}^n$ is a Hilbert space with respect to an inner product to be defined by you. [4]

   (c) Prove that $C[a,b]$ with respect to supnorm is not a Hilbert space. [2]

6. (a) Show that every inner product space is a normed linear space. [3]

   (b) State and prove Riesz representation theorem for a bounded linear functional over a Hilbert space. [5]

   (c) Show that closure of convex set in a normed linear space is convex. [2]

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M. Sc. Examination-2023

Semester-II

Mathematics

Course : MMC-22 (Topology)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any **four** questions

1. (a) Show that for any set X, $|\mathcal{P}(X)| = 2^{|X|}$ [3]
   (b) State and prove Schroder-Bernstein theorem for cardinal numbers. [4]
   (c) Define a maximal element of a poset. Also show that it is not unique. [3]

2. (a) Define a metrizable space. Prove that the space $X = \{a, b\}$ with the topology $\tau = \{X, \emptyset, \{a\}\}$ is not metrizable. [3]
   (b) Show that $(A \cup B)' = A' \cup B'$ where A is derived set of A. [2]
   (c) Define Kuratowski closure operator c on a set X, Also show that it induces a topology on X so that $c(A) = \overline{A}$ for all $A \subset X$. [5]

3. (a) Show that the following are equivalent for a function f from one topological space to other:
   i) The function f is continuous.
   ii) The inverse image of every open set under f is open. [$3\frac{1}{2}$]
   (b) Define an open map, give an example with justification of an open map which is not continuous. [$2\frac{1}{2}$]
   (c) Define a closed map. Prove that a map f is closed if and only if $\overline{f(A)} = f(\overline{A})$ for a set A of a topological space. [1+3]

4. (a) Show that every 2nd countable space is 1st countable, separable, and Lindelof. [$1\frac{1}{2}+1\frac{1}{2}+2$]
   (b) Give an example with justification of a 1st countable space which is not 2nd countable. [2]
   (c) Show that $\mathbb{R}$ with lower limit topology is Lindelof. [3]

5. (a) Define T_1 and T_2 spaces. Give an example with justification of a T_1 space which is not T_2. [2+2]
   (b) Prove that each one point set is closed in a T_1 space. [2]
   (c) Show that a space is regular if and only if $\forall x \in X$ and $\forall$ open set $V \subset X$ containing x, $\exists$ an open set V_x containing x such that $\overline{V_x} \subset V$. [4]

6. (a) When is a topological space is said to be normal ? Show that every compact T_2 space is normal. [4]
   (b) Define FIP property. Prove that a space X is compact if and only if each family of closed sets in X with FIP property has a non empty intersection. [4]
   (c) Consider $\mathbb{R}$ with co-finite topology. Show that every subset of $\mathbb{R}$ is compact. [2]

M. Sc. Examination-2023

Semester-II

Mathematics

MMC-23

(Abstract Algebra)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.

Notations and symbols have their usual meanings.

Answer ***any four*** questions.

$\mathbb{Z}_{12}=\mathbb{Z}_3\times\mathbb{Z}_4$

1. (a) Show that $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq S_3$. [3]

   (b) Can the cyclic group Z_{12} be expressed as an internal direct product of two nontrivial subgroups? Justify your answer. [3]

   (c) Let H be a subgroup of group G of index n. If H does not contain any nontrivial normal subgroups of G, show that H is isomorphic to a subgroup of S_n. [4]

2. (a) Let G be a group of order pn, p a prime be such that $p > n$. Show that [illegible] a normal subgroup of G. [3]

   (b) Write down the class equation of the symmetric group S_5. [4]

   (c) For any prime p, prove that any group of order p^2 is either cyclic or a direct product of cyclic groups. [3]

3. (a) Show that any simple group of order 60 is isomorphic to A_5. [4]

   (b) Classify all groups of order 14. [3]

   (c) How many groups of order 35 are there up to isomorphism? [3]

$\frac{\mathbb{R}[x]}{\langle x\rangle} \simeq \mathbb{R}$

$\phi : \mathbb{R}[x,y] \to \mathbb{R}[y]$

4. (a) Is $\frac{\mathbb{R}[x,y]}{<x>}$ a field? Justify your answer. [3]

   (b) Show that every ED is a PID. [4]

   (c) Let $(E, +, \cdot, v)$, be an ED. If a, b are associates then show that $v(a) = v(b)$. [3]

5. (a) Show that every prime ideal in PID is a maximal ideal. Give an example with proper reasons that a nonzero prime ideal in a ring may not be a maximal ideal. [3+1]

   (b) In the ring $\mathbb{Z}[i\sqrt{5}]$ find GCD of $6(1+i\sqrt{5})$ and $3(1+i\sqrt{5})(1-i\sqrt{5})$, if exists. [3]

   (c) Show that the polynomial $x^5 + x^2 + 1$ is irreducible over $\mathbb{Z}_2$ and hence $\frac{\mathbb{Z}_2[x]}{<x^5+x^2+1>}$ is a field. [2+1]

6. (a) State Eisenstein's irreducibility criterion and use it to check the irreducibility of the polynomial $x^6 + x^3 + 1$ over $\mathbb{Z}$. [4]

   (b) Is $\mathbb{Z}[i]$ a FD? Give reasons in support of your answer. Yes / no ? [3]

   (c) If $f(x)$ is an irreducible polynomial over the field of real numbers $\mathbb{R}$ then show that either $f(x)$ is linear or quadratic. [3]

M. Sc. Examination-2023
Semester-II
Mathematics
Course: MMC-24
(Classical Mechanics)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. The $Oxyz$ coordinate system is rotating with angular velocity $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$. Here $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along the axes of the coordinate system $Oxyz$. Derive the relations among true and apparent velocities and accelerations of a moving particle. Hence obtain the equation of motion of a particle of mass m with reference to $Oxyz$ moving under the influence of a physical force $\vec{F}$. [3+4+3]

2. Derive the equation of motion of a particle relative to a frame of reference fixed at a point of co-latitude λ on the earth's surface (state clearly the omission of terms during your derivation, if any). An object is dropped from a height h above a point with latitude θ on the earth's surface. If the air resistance is neglected, find the point on the earth's surface where the particle will hit. Does the point of hitting on the earth's surface remain the same if the earth's rotation is neglected? [4+5+1]

3. When a rigid body is said to be a symmetric rigid body? Define Euler's angles and interpret them in the context of the rotational motion of a symmetric rigid body about a point fixed on its axis of symmetry. Establish the correspondence between the component of angular velocity $\vec{\omega}(t)$ and Euler's angles and their derivatives. Hence, find the expression for the angular momentum $\vec{\Omega}(t)$ and the kinetic energy due to rotational motion about the origin of principal axes of a symmetric rigid body. [1+3+3+3]

4. Write down the expressions for the kinetic energy (T) and potential energy (V) for the rotational motion of a heavy symmetric rigid body rotating about a point fixed on its axis of symmetry in terms of Euler angle variables. Hence write down the expression for Lagrangian for the motion. Derive the equation for changes of Euler angle variables from this Lagrangian. [4+2+4]

5. State D'Alembert's principle and derive Lagrange's equation of motion from it. Define the generalized coordinate describing the motion of a particle of mass m suspended from a point in the ceiling, oscillating in a vertical plane around its stable equilibrium. Derive the kinetic (T) and the potential (V) energy for the motion. Hence, find the Lagrangian of the system and the particle's equation of motion. [1+4+2+2+1]

6. Define the Poisson bracket and state some of its essential properties. Write down the expressions for fundamental Poisson brackets. Prove the Jacobi identity for the Poisson bracket. Derive the equation of motion of any dynamical variable in the phase space.[1+2+1+4+2]

Use separate answer script for each unit

M. Sc. Examination-2023

Semester-II
Mathematics
Paper : MMC-25 (New Syllabus)
(Solid Mechanics and Dynamical System)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I [Solid Mechanics (Marks: 20)]

Answer *any two* questions.

1. (a) Write down the expressions for Lagrangian and Eulerian infinitesimal strain tensor. Prove that, in case of infinitesimal deformation, the distinction between them disappears. [2+3]

   (b) Obtain the relation between strain vector and strain tensor. A body undergoes deformation

$$
\begin{aligned}
x_1 &= \sqrt{2}X_1 + \frac{3}{4}\sqrt{2}X_2 \\
x_2 &= -X_1 + \frac{3}{4}X_2 + \frac{\sqrt{2}}{4}X_3 \\
x_3 &= X_1 - \frac{3}{4}X_2 + \frac{1}{4}\sqrt{2}X_3
\end{aligned}
$$

   Find the direction after deformation of a line element with initial direction ratios 1:1:1. [2+3]

2. (a) Derive stress equation of equilibrium. [5]

   (b) The stress tensor at a point is given by
   $\tau_{ij} =$

$$
\begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}
$$

   Determine principal stresses and the maximum shear stress. [5]

$(x+1)^2 - 5$
$x^2+2x+1-5$
$x^2+2x-4.$

3. (a) Prove that the extremum values of normal stress at a point of a continuum are principal stresses. [5]

   (b) State the principle of conservation of mass in spatial method and hence obtain the equation of continuity. [2+3]

Unit-II [Dynamical Systems (Marks: 20)]

Answer *any two* questions.

1. (a) Consider the map $f(x) = \lambda x - x^3$, where λ is a real parameter. Determine the fixed points and their stabilities for different values of the parameter λ. [2]

   (b) Consider the Hénon map $H_{a,b}(x, y) = (a - x^2 + by, x)$, where a, b are the two real parameters. Prove that the map has a periodic 2-orbit iff $a > \frac{3}{4}(1-b)^2$. [4]

   (c) State Poincaré-Bendixson theorem. Show that the system given by $\dot{x} = x - y - x(x^2 + 2y^2)$, $\dot{y} = x + y - y(x^2 + 2y^2)$ has at least one periodic orbit. [4]

2. (a) Consider the map $f(x) = x^2 + k, x \in \mathbb{R}$, where k is a real parameter. Show that the map f has a period-2 orbit if $k < -\frac{3}{4}$. Also show that the orbit is stable if $-5/4 < k < -3/4$. [3]

   (b) Find parameter (r) values so that the Logistic map $f_r(x) = rx(1-x)$, $x \in [0, 1]$ has a superstable fixed point and a superstable 2-cycle. [3]

   (c) State Bendixson's Negative Criterion. Show that the system

$$\dot{x} = -y + x(x^2 + y^2 - 1), \quad \dot{y} = x + y(x^2 + y^2 - 1)$$

   has no closed orbits with center at $(0, 0)$ and radius $\frac{1}{\sqrt{2}}$. [2+2]

$x = \frac{1 \pm \sqrt{1-4K}}{2}$

3. (a) Show that $\left\{\frac{1}{3}, \frac{2}{3}\right\}$ is a periodic 2-cycle of the Euler shift map $S(x)$.

(b) Define and plot the Tent map $T(x)$. Also find the fixed points of $T^2(x)$. [2-

(c) State the conditions under which the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ has a unique stable limit cycle surrounding the origin of the phase plane. Show that the equation $\ddot{x} + \beta(x^2 - 1)\dot{x} + x = 0$ has a unique stable limit cycle if $\beta > 0$. [2+2]

M.Sc. Examination, 2023

Semester-II

Mathematics

Course: MMC-26 (New & Old Syllabus)

Numerical Analysis

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Show that $f[x_0, x_1, \cdots, x_n] = \beta^{-n} F[y_0, y_1, \cdots, y_n]$ under the linear transformation $x = \alpha + \beta y$, where $x_i = \alpha + \beta y_i$, $i = 0, 1, 2, \cdots, n$ and $f(\alpha + \beta y) = F(y)$. [2]

   (b) Defining a Hermite interpolating polynomial, show that it is the unique polynomial of least degree agreeing with f and f' at the nodes $x_0, x_1, \cdots, x_n$, when f is the function to be approximated. [1+3+2]

   (c) What do you mean by clamped cubic spline? Explain with a suitable example. [2]

2. (a) How Chebyshev polynomials minimize the error in Lagrange interpolation? Explain in details. [5]

   (b) The 2nd degree polynomial $f(x) = a + bx + cx^2$ is obtained from the condition $d = \sum_{i=m}^{n} [f(x_i) - y_i]^2 =$ minimum where (x_i, y_i), $i = m(1)n, m < n$, are given real numbers. Putting $X = x - \xi$, $Y = y - \eta$, $X_i = x_i - \xi$, $Y_i = y_i - \eta$, we determine $F(X) = A + BX + CX^2$ from the condition $D = \sum_{i=m}^{n} [F(X_i) - Y_i]^2 =$ minimum. Show that $F(X) = f(x) - \eta$. Also, derive an explicit formula for $F'(0)$ expressed in terms of Y_i when $X_i = ih$ and $m = -n$. [5]

3. (a) What do you mean by Romberg's quadrature? Derive Romberg's quadrature formula for evaluating $I = \int_a^b f(x)dx$ by Simpson's one-third rule. Also mention the stopping criterion. [1+4+1]

   (b) Evaluate the integral $I = \int_{-1}^{1} (1 - x^2)^{\frac{5}{2}} \cos 2x \, dx$ by using the Gauss-Legendre three-point formula. [4]

4. (a) Determine the largest eigenvalue and the corresponding eigenvector of the following matrix correct to three decimal places using the power method. [5]

$$M = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

   (b) Transform the following matrix to tridiagonal form by Givens method. [5]

$$A = \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & 2 \\ \sqrt{2} & -\sqrt{2} & -1 & \sqrt{2} \\ \sqrt{2} & -1 & \sqrt{2} & \sqrt{2} \\ 2 & \sqrt{2} & \sqrt{2} & -3 \end{pmatrix}$$

5. (a) Establish Adams-Moulton method of order four for solving numerically the well-posed initial value problem $\frac{dy}{dx} = f(x, y)$, $y(a) = y_0$ in a finite interval $[a, b]$. [5]

   (b) Compute $y(1.2)$, from the differential equation $\frac{dy}{dx} = 0.05x^{-2} - y^2, y(1) = 1$, taking $h = 0.05$, by Modified Euler's method, correct to four decimal places. [5]

6. (a) What do you mean by equilibrium problem? Explain with a suitable example. [2]

   (b) Find a forward difference approximation of $O(\Delta y)$ for $\frac{\partial^4 u}{\partial y^4}$. [3]

   (c) What do you mean by stability of a finite difference scheme? [2]

   (d) Derive the Crank-Nicholson (C-N) scheme for approximating the parabolic equation $u_t = \alpha u_{xx}$ at the $(i, n)^{\text{th}}$ point with α being positive constant. [3]