M. Sc. Examination-2023

Semester-IV

Mathematics

Course: MMC-41(New)

(Differential Geometry and Manifold Theory)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Define smooth manifold of dimension n, $n \in \mathbb{N}$. What do you mean by local coordinates for a manifold? Find an atlas with appropriate charts for $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ which provides a smooth manifold structure on $\mathbb{S}^1$ mentioned above of some dimension to be determined by you. [4+5+1]

2. Introduce the concept of the tangent space $T_p\mathcal{M}$ at a point p on a manifold $\mathcal{M}$. Derive a basis for $T_p\mathcal{M}$ in terms of the local coordinates $(x^1, x^2, \ldots, x^m)$ of $\mathcal{M}$. Define the Lie bracket (commutator) of two tangent vectors in $T_p\mathcal{M}$ and verify whether the tangent space $T_p\mathcal{M}$ forms a Lie algebra with the bilinear map as the Lie bracket. Define vector fields on $\mathcal{M}$ and its action on smooth functions $C^\infty(U)$, where U is some neighborhood of $\mathcal{M}$. Prove that each vector field $\mathbf{X} \in \mathcal{X}(U)$ is a linear derivation on $C^\infty(U)$. [2+2+3+2+1]

3. Define differential forms and wedge product of forms on the manifold $\mathbb{R}^3$. Introduce the notion of exterior derivative of differential forms in $\Lambda_p(\mathbb{R}^3)$. Prove the formula $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \;\; \forall \alpha \in \Lambda_p(\mathbb{R}^3), \beta \in \Lambda_q(\mathbb{R}^3)$. When a form is said to be exact? Prove that every exact form is closed in $\mathbb{R}^3$. [2+2+3+1+2]

4. What do you mean by the first (I) and the second (II) fundamental forms for a smooth surface $\mathcal{S} \subset \mathbb{R}^3$? Give their geometrical significance. Obtain I and II for the (sphere) surface $\vec{r}(u,v) = (a \sin u \cos v, a \sin u \sin v, \, a \cos u)$. [2+2+3+3]

5. Define normal and geodesic curvatures of curves on a smooth surface $\mathcal{S} \in \mathbb{R}^3$. Obtain the expression for normal curvature in terms of the coefficients of fundamental forms. Obtain formulas for Gaussian and mean curvatures from the expression just obtained by you. Compute the Gaussian and mean curvatures for the sphere $\vec{r}(u,v) = (\sin u \cos v, \sin u \sin v, \cos v)$. [2+3+3+2]

6. State the definition of covariant derivative $\vec{\nabla}_{\mathbf{X}}(\mathbf{Y})$. Introduce Riemann-Christoffel curvature tensor R^l_{ijk}. State and prove Gauss's Theorema Egregium in terms of R^l_{ijk}. State Bianchi's identity and define Ricci tensor. When a Riemannian space is said to be flat? [1+2+4+1+1+1]

M. Sc. Examination-2023

Semester-IV
Mathematics
Course: MMC-41(Old)
(Differential Geometry and Manifold Theory)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Define normal section of a smooth surface $\mathfrak{S}$ at a point $P(x^1(u^1,u^2), x^2(u^1,u^2), x^3(u^1,u^2))$ on the surface in the direction of a curve $\Gamma(t)$ $\left(x^1(u^1(t),u^2(t)), x^2(u^1(t),u^2(t)), x^3(u^1(t),u^2(t))\right)$ on it. Hence find the normal section to the surface $\mathfrak{S} = \left\{(x^1,x^2,x^3) : (x^1)^2 + (x^2)^2 - (x^3)^2 = 1\right\}$ at the point $Q(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ along a curve whose tangent is directed along $\mathbf{w}_1 = (1,0,0)$ at Q. Find curvature and torsion to this curve. [2+4+4]

2. Introduce the concept of the first and the second fundamental forms on a smooth surface $\mathfrak{S}$. Find the first and the second fundamental forms on the surfaces $\mathfrak{S}_1 = \{(\cos u^1, \sin u^1, u^2),\ \ u^1 \in [0,2\pi], u^2 \in \mathbb{R}\}$. [5+5]

3. Explain how the Christoffel symbols are involved in the second derivative of the position vector of a point on a curve on the smooth surface with respect to the parameters involved. Find the expression for components of these symbols for the surface $\mathfrak{S} = \{(\sin u \cos v,\ \sin u \sin v,\ \cos u) : v \in [0,2\pi], u \in [0,\pi]\}$. [2+3+5]

4. Define normal curvature in the direction $\mathbf{v}$ at a point Q on a surface $\mathfrak{S}$. Introduce principal curvatures and principal directions. Prove that principal directions on a surface are orthogonal. Define Gauss and mean curvature of a surface. [1+4+5]

5. Prove the formula $K = \frac{L}{g}$. Find the parameter dependence of Gauss curvature at a point (u,v) on the surface $\mathfrak{S} = (u, v, f(u,v))$. [5+5]

6. Derive the equation for geodesic on a surface, in general. Hence find geodesics on a cylindrical surface joining $(\rho,0,1)$, $(\rho,0,5)$ and $(\rho,0,1)$, $(\rho,\pi,1)$ $(\rho = \text{const.})$. Interpret the nature of (geodesic) curves. [5+4+1]

M. Sc. Examination-2023

Semester-IV
Mathematics
Paper: MMC 42
(Operations Research)

Time: 3 Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Why is the revised simplex method superior to the regular simplex method? Explain briefly. [2]

   (b) Solve the following linear programming problem (LPP) by revised simplex method:

$$\begin{aligned} \text{Minimize } f(x_1,x_2) &= 0.8x_1 - 6x_2 \\ \text{subject to } -x_1+x_2 &\geq 5, \\ x_1 - x_2 &\geq 2, \\ x_1, x_2 &\geq 0. \end{aligned}$$

   Provide an explanation for the characteristics of the optimal solution. [7+1]

2. (a) What kind of purpose does sensitivity analysis serve in operations research? [2]

   (b) Consider the following LPP:

$$\begin{aligned} \text{Maximize } f(x_1,x_2) &= 3x_1 - x_2 \\ \text{subject to } 2x_1+x_2 &\geq 2, \\ x_1 + 3x_2 &\leq 3, \\ x_2 &\leq 4, \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

   (i) Find the ranges of the objective function coefficient c_1 and c_2 of the variables x_1 and x_2 such that the optimality is unaffected.
   (ii) Will the optimal solution to the LPP remain optimal, if the requirement vector $[2,3,4]^T$ changes to $[6,2,1]^T$? Justify your answer. [4+4]

3. (a) Differentiate between integer linear programming problem (ILPP) and LPP using real-world issues. [2]

   (b) Find the optimum integer solution of the following LPP by Gomory's cutting plane method / branch and bound method:

$$\begin{aligned} \text{Maximize } f(x_1,x_2) &= 2x_1 + 1.7x_2 \\ \text{subject to } 4x_1+3x_2 &\leq 7, \\ x_1 + x_2 &\leq 4, \\ x_1, x_2 &\geq 0 \text{ and are integers.} \end{aligned}$$

   [8]

4. (a) State Kuhn-Tucker necessary and sufficient conditions for a non-linear programming problem (NLPP) with maximization type objective function and with all 'less than or equal to' type constraints. Hence solve the following NLPP:

$$\begin{aligned} \text{Maximize } f(x_1,x_2) &= 8x_1 + 10x_2 - x_1^2 - x_2^2 \\ \text{subject to } 1.5x_1+x_2 &\leq 3, \\ 3x_1 + 2x_2 &\leq 6, \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

   [2+4]

(b) Solve the following NLPP graphically:

$$\begin{aligned} \text{Maximize } f(x_1, x_2) &= 2x_1 + 3x_2 - x_1^2 \\ \text{subject to } x_1 + 2x_2 &\leq 4, \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

[4]

5. (a) Write down the basic assumptions of an Economic Order Quantity (EOQ) purchase model with instantaneous replenishment and with shortages. Hence, show that the mathematical expression for the optimal lot size is $Q^* = \sqrt{\frac{2Dc_s(c_1+c_2)}{c_1 c_2}}$; where D, c_s, c_1 and c_2 respectively represents the demand / period, ordering cost / order, carrying cost / unit / period and shortage cost / unit / period. [6]

(b) The demand of a bought out item in a store is 24000 units per year. The carrying cost is Rs. 10 per unit per year and the ordering cost is Rs. 900 per order. The shortage cost is Rs. 30 per unit per year. Find the EOQ and the corresponding number of orders per year, the maximum inventory and maximum shortage quantity. [4]

6. (a) For the queuing model $M/M/1 : N/FIFO$, having Poisson input, exponential service, single service channel, finite capacity of the system, first in first out queue discipline; find the probability of having 0 customer in the system and average queue length of the system. [6]

(b) A weighing station has single weighing bridge. The arrival rate of the vehicles coming to the weighing station follows Poisson distribution and it is 45 vehicles per hour. The service rate also follows Poisson distribution and it is 55 vehicles per hour. In front of the weighing bridge, the waiting space is sufficient for a maximum of 10 vehicles. Find the following:

(i) Average waiting number of vehicles in the queue in front of the weighing bridge as well as in the weighing station.

(ii) Average waiting time per vehicle in front of the weighing bridge as well as in the weighing station. [2+2]

Use separate answer script for each unit

M. Sc. Examination-2023

Semester-IV
Mathematics
Elective Course: MME-41 (Applied Stream)
(Electromagetic Theory and Programming in Matlab)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Electromagnetic Theory)

Answer ***any two*** questions.

1. (a) State Coulomb's law in electrostatics. Using Gauss' law in electrostatics, calculate the electric field outside a uniformly charged sphere of radius R and the total charge Q. [1+3]

   (b) For a point charge configuration in a spatially uniform electric field, define an electric dipole and torque. Calculate the potential energy of an electric dipole placed in an external electric field. Hence calculate the net force acting on the dipole. [2+3+1]

2. (a) Derive the equation of continuity for a steady state current of electric charges. What is Ohm's law? Use it to show that for a steady state current, the continuity equation reduces to the Laplace equation for the electrostatic potential. [2+2+1]

   (b) Define the magnetic force acting on a charged particle of charge q. State and explain the Amperé's circuital law. What are the shortcomings of this law? [1+2+2]

3. (a) Explain the phenomenon of electromagnetic induction and hence establish the differential form of Faraday's law. [2+3]

   (b) State and prove the Poynting's theorem. What is Poynting's vector? [1+3+1]

Unit-II (Programming in Matlab)

Answer question no. 1 and ***any one*** from the rest.

1. Choose the correct alternative (*any five*).

   (a) What does MATLAB stand for? (i) Math Laboratory (ii) Matrix Laboratory (iii) Mathworks (iv) Mathematics Lab. [2]

   (b) Which of the following is not a predefined variable in MATLAB? (i) pi (ii) inf (iii) i (iv) gravity. [2]

   (c) In MATLAB $--$ command is used to add text label to x-axis. (i) XAxis (ii) xaxis (iii) Xlabel (iv) xlabel. [2]

   (d) Which type of analysis is not done by MATLAB software? (i) Numerical computing (ii) Algebraic solutions (iii) Plant planning analysis (iv) Dynamic system simulations. [2]

   (e) Which of the following is not a valid plotting command in MATLAB? (i) Figure (ii) Prod (iii) Print (iv) Plot. [2]

   (f) Which of the following correctly defines x, y, and z as symbols? (i) sym (x,y,z) (ii) syms x y z (iii) syms x, y, z (iv) sym x, y, z. [2]

   (g) Which of these is the way to access the first element in a vector named v (assuming that there is at least one element in the vector)? (i) $v(0)$ (ii) $v(1)$ (iii) v (iv) $v(:,0)$. [2]

(h) To add a comment to an mfile, the MATLAB command is (i) % (ii) & (iii) comment (' ') (iv) #. [2]

(i) The output of Cat=['Cat''Dog'] is (i) 'Cat'Dog' (ii) Cat Dog (iii) CatDog (iv) Cat & Dog. [2]

2. (a) If $A = [1\ 3\ 5; 2\ 4\ 6; 7\ 8\ 9]$ is a matrix, then write the outputs of (i) $A(3,2)$, (ii) $A(2:3,3)$, (iii) $A(2,:)$, and (iv) $A(:,3)$. [6]

(b) Write the Matlab command to find the roots of a polynomial equation $5x^5 - x^2 + 7 = 0$. [2]

(c) Write a MATLAB program to plot the function $f(x) = e^{-x}\sin x$, $-\pi \le x \le \pi$. [2]

3. Write a MATLAB program to solve the coupled nonlinear equations: $x = y + \cos(x+y)$, $y = x + \sin(xy)$. [10]

Use separate answer script for each unit

M. A./M. Sc. Examination-2023

Semester-IV
Mathematics
Elective Course: MME-41 (Pure Stream)
(Galois Theory-II and Algebraic Topology)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 20)
(Galois Theory-II)

Answer *any two* questions.

1. (a) Show that every nonconstant polynomial over a field of characteristic 0 is seperable. [3]
   (b) Give an example of a inseparable polynomial over a field with finite characteristic. Give reasons to justify your answer. [2]
   (c) Let F/K be a finite field extension of characteristic $p > 0$ such that $p \nmid [F : K]$. Prove that F/K is not separable. Give an example of such extension with proper justification. [3+2]

2. (a) Find the Galois group of the polynomial $(x^2 - 2)(x^2 - 3)$ over $\mathbb{Q}$. Which group is isomorphic to this group upto isomorphism? Give reasons to justify your answer. [4+2]
   (b) Find the generator of the Galois group $G(F_{p^3}/F_p)$. [4]

3. (a) Prove that a finite extension F/K is Galois extension if and only if $G(F/K) = [F : K]$. [4]
   (b) Let K be a field and n be a positive integer. Suppose that characteristic $K \nmid n$. Let R be set of all $n-$th roots of unity in the splitting Field F of $x^n - 1$ over K. Then show that $\xi \in F$ is a primitive $n-$th root of unity if and only if $R =< \xi >$. [3+3]

Unit-II (Full Marks: 20)
(Algebraic Topology)

Answer *any two* questions.

4. (a) Let X be a topological space and let f, g, h be three paths in X from x_1 to x_2, from x_2 to x_3, from x_3 to x_4 respectively. Prove that $[f] * ([g] * [h]) = ([f] * [g]) * [h]$. [5]
   (b) What is a contractible space? Explain with examples. Examine if the disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$ is a contractible space. Give an example of a non-contractible space. [1+1+2+1]

5. (a) What do you mean by a simply connected space? Show that a path connected space X is simply connected if and only if any two paths in X with the same initial and final points are path homotopic. [1+5]
   (b) Let $X = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + (y - 2)^2 = 4\}$ and let $Y = X \setminus \{1, 4\}$. Compute $\pi_1(Y, (3, 2))$ and $\pi_1(Y, (-1, 2))$. [2+2]

6. (a) Let X and Y be two topological spaces. Show that a covering map $p : X \longrightarrow Y$ is an open map. Is the converse true? Justify your answer. [3+2]
   (b) Explain the path lifting property for a covering space. Prove that the fundamental group of S^1 is $(\mathbb{Z}, +)$. [1+4]

M. Sc. Examination-2023

Semester-IV

Mathematics

Optional Paper: MM0-41(P11/A13)

(Weak Formulation of Elliptic Partial Differential Equations)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Define test functions and distributions. Verify whether $e^{-|x|}$ may be regarded as a test function. State the difference between regular and singular distributions with an illustration. Prove that two continuous functions that produce the same regular distributions are identical. Define the derivative of a distribution. Find the derivative of $|x|$ if exists. [1+1+3+2+1+2]

2. Define δ-distribution. Express $\delta((x-a)(x-b))(a<b\in\mathbb{R})$ as a sum of two delta functions. Simplify the distribution $(x^2-1)\delta(x^2-1)$ and $e^x\delta(x-a)$. Prove that $g(x)\delta'(x-a)=g(a)\delta'(x-a)-g'(a)\delta(x-a)$. [1+2+4+3]

3. Define functions of rapid decay, slow growth and tempered distribution. Prove that the Fourier transform of a function of rapid decay is also a function of rapid decay. Verify the formula $\mathcal{F}[t'](u)=iu(\mathcal{F}[t])(u)$. [3+4+3]

4. Prove that if $f(x)$ is a classical function and $f(x)+\sum_{k=0}^{n}a_i\delta^{(i)}(x-\xi)=0$ on $\mathbb{R}$, then $f(x)=0$ and $a_0=\cdots=a_n=0$. Remove x^5 from $x^5\delta^{(3)}(x)$. [6+4]

5. Define the classical, weak and distributional solutions of an ordinary differential equation. Find the general solution of $x^k\frac{dy}{dx}=0$. State the nature of the solution with justification. Prove that $t(x)=E*\tau$ may be regarded as the solution of $\frac{d^2t}{dx^2}=\tau(x)$ for $\tau\in\mathcal{D}'$ where E is the fundamental solution of a differential equation to be determined by you. [3+3+4]

6. When a linear operator involving second-order derivatives is called V-elliptic? State Riesz representation theorem and Lax-Milgram lemma. Explain the in-depth relationship between the theorem and the lemma mentioned above clearly. Exercise Lax-Milgram lemma on the problem $u''(x)+u(x)=f(x)$ on $\Omega=(0,1)$ with $u(0)=0,u(1)=0$. [2+3+2+3]

M.A./M.Sc. Examination-2023
Semester-IV
Mathematics
Optional Course: MMO 41 (A8)
(Mathematical Pharmacology-II)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Write down the governing equations for a two-compartment pharmacokinetics model with drug absorption with a first-order elimination kinetics from the central and peripheral compartments. Obtain the time-dependent drug concentration in each compartment. [3+7]

2. What is meant by flow-limited model? Consider a two-comparment flow-limited model and obtain the unsteady drug concentrations therein. [3+7]

3. What is ideal stirred-tank model? Assuming suitable initial conditions, obtain the expressions for drug concentration in a two-compartment organ in an ideal stirred-tank model. [3+7]

4. Using appropriate initial and boundary conditions, obtain the total amount of drug released from the matrix delivery system. [10]

5. Asuming the rate of release of drug from a matrix depends on the rate of diffusion through polymer matrix, obtain the drug concentration, assuming suitable initial and boundary conditions, in an implanted polymer matrix at any instant t. [10]

6. Write short notes on
   (i) Compartment model
   (ii) Physiological model. [5+5]

M. Sc. Examination-2023

Semester-IV

Mathematics

Paper: MMO 41 (A5)

(Dynamics of Ecological System-II)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Examine the effect of discrete time delay on the dynamics of the following predator-prey system:

$$\begin{aligned}\frac{du}{dt} &= \frac{au}{1+fv} - bu - cu^2 - \frac{\alpha uv}{\beta + u},\\ \frac{dv}{dt} &= \frac{\gamma u(t-\tau)v}{\beta + u(t-\tau)} - \delta v,\\ u(t=0) = u_0(t) &> 0,\ v(t=0) = v_0(t) > 0,\ t \in [-\tau, 0],\end{aligned}$$

where $\tau > 0$ represents the reaction time of the predation and f is the level of fear that causes the prey to exhibit anti-predator behaviours. [10]

2. What benefits may a dimensionless model offer? Construct a dimensionless model by appropriately transforming the modified Hastings-Powell system shown below:

$$\begin{aligned}\frac{dX}{dT} &= X\left[R_0\left(1-\frac{X}{K}\right) - \frac{M}{X+B}\right] - \frac{A_1XY}{B_1+X},\\ \frac{dY}{dT} &= \frac{C_1A_1XY}{B_1+X} - \frac{A_2YZ}{B_2+Y} - D_1Y - H_1Y^2,\\ \frac{dZ}{dT} &= \frac{C_2A_2YZ}{B_2+Y} - D_2Z - H_2Z^2,\\ X(T=0) &= X_0 \geq 0,\ Y(T=0) = Y_0 \geq 0,\ Z(T=0) = Z_0 \geq 0,\end{aligned}$$

where, $X(T)$, $Y(T)$ and $Z(T)$ respectively represent the population sizes of prey, middle predator and top predator at time T; M and B are the Allee constants and $K > B$, $0 < C_1, C_2 < 1$. Hence, show that the dimensionless model is uniformly bounded. [(2+5)+3]

3. Consider the following classic three-species food chain model developed by Hastings and Powell:

$$\begin{aligned}\frac{du}{dt} &= ru\left(1-\frac{u}{K}\right) - \frac{c_1a_1vu}{b_1+u},\\ \frac{dv}{dt} &= \frac{a_1uv}{b_1+u} - \frac{a_2vw}{b_2+v} - d_1v,\\ \frac{dw}{dt} &= \frac{c_2a_2vw}{b_2+v} - d_2w,\ 0 < c_1^{-1} < 1,\ 0 < c_2 < 1,\\ u(0) &> 0,\ v(0) > 0,\ w(0) > 0,\end{aligned}$$

where the populations of prey, middle predator and top predator at time t are represented by u, v and w, respectively. Put down your thoughts on the following concerns at the coexistence equilibrium point:
(i) Hopf-bifurcation,
(ii) Chaotic dynamics. [6+4]

4. Obtain the necessary and sufficient conditions for diffusion-driven instability of the following two species interaction:

$$\begin{aligned}\frac{\partial u}{\partial t} &= f_1(u,v) + D_1\nabla^2 u,\\ \frac{\partial v}{\partial t} &= f_2(u,v) + D_2\nabla^2 v,\\ u(0,x) &\geq 0,\ v(0,x) \geq 0,\end{aligned}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}$ and all the system parameters are positive. [10]

5. A reaction-diffusion model is given by

$$\begin{aligned}\frac{\partial u}{\partial t} &= \alpha(\beta - u + u^2 v) + \nabla^2 u,\\ \frac{\partial v}{\partial t} &= \alpha(\gamma - u^2 v) + \delta\nabla^2 v,\\ u(0,x) &\geq 0,\ \ v(0,x) \geq 0,\end{aligned}$$

where u and v are morphogen concentrations at (x,t); $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}$.
(i) Explain the model and find the non-zero homogeneous steady state.
(ii) Show that the condition of stability (without diffusion) is $\gamma - \beta - (\gamma + \beta)^3 < 0$.
(iii) Linearize the system about the non-zero steady state and find the condition for diffusive instability. [2+3+5]

6. Analyze the spatial pattern initiation for the following Schnäkenberg system including diffusion:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \gamma(\alpha - u + u^2 v) + \nabla^2 u,\\ \frac{\partial v}{\partial t} &= \gamma(\beta - u^2 v) + \delta\nabla^2 v,\\ u(0,x,y) &> 0,\ \ v(0,x,y) > 0,\\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0,\ \ (x,y) \in \partial\Omega,\end{aligned}$$

where u and v represent concentrations of morphogens at (x,y,t); $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; n is the outward unit normal vector of the smooth boundary $\partial\Omega$ of the reaction-diffusion domain Ω. [10]

M.Sc. Examination - 2023

Semester-IV

Mathematics

Course: MMO-41 (A-03)

(Computational Fluid Dynamics - II)

Time: Three Hours Full Marks: 40

Questions are of values as indicated in the margin

Notations and symbols have their usual meanings

Answer ***any four*** questions

1. Discuss the spread of a plane free jet considering the component of velocity in the $x-$direction, $u = Ax^{p-q}f'(\eta)$ where $\eta = y/x^q$. Show that the mass flux varies as $x^{1/3}$ where x is the distance measured downstream from the slit of the jet. [10]

2. (a) Derive any one form of Von Karman's integral equation. [5]

   (b) Determine the displacement thickness and momentum thickness for the laminar boundary layer on a flat plate for which the velocity distribution is given by the relation

$$u/U = 2(y/\delta) - 2(y/\delta)^2 + (y/\delta)^4$$

[5]

3. (a) Find the first and second order accurate vorticity value at the boundary flat plate. [5]

   (b) When

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$$

   is written in the transformed (ξ, η) space, can it be recast in strong conservation form? Give justification in support of your answer. [5]

4. Describe MacCormack's technique for the solution of unsteady two dimensional Euler equations using the calculation of density. [10]

5. Derive the sixth order compact difference scheme for the Laplace equation. [10]

6. Find the high-order compact $O(h^3)$ corner boundary conditions for upper-right and lower-left corners of a rectangular cavity while solving Navier-Stokes equations for viscous, incompressible flow using stream-function vorticity form using $M \times N$ grid. [10]

M. Sc. Examination-2023

Semester-IV

Mathematics

MMO 41 : Optional Paper

P-10 : Rings and Modules-II

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Let R be a unitary ring and M be an abelian group. Prove that M is an R-module if and only if there is a 1-preserving ring homomorphism from R into $End(M)$. [3]

   (b) Let A and B be two submodules of an R-module M. Show that the sequence $0 \longrightarrow M/(A\cap B) \longrightarrow M/A\times M/B \longrightarrow M/(A+B) \longrightarrow 0$ is exact. Deduce that $(A+B)/(A\cap B) \simeq (A+B)/A \times (A+B)/B \simeq B/(A\cap B) \times B/(A\cap B)$. [4]

   (c) Let M be an R-module. Show that M can be considered as an $\overline{R} = R/Ann_R(M)$-module such that $Ann_{\overline{R}}(M) = 0$. [3]

2. (a) Let $f : M \longrightarrow N$ be an R-morphism. For every submodule A of M show that $f^{-1}(f(A)) = A + \ker f$. [3]

   (b) Give example to show that an one-to-one R-morphism in the category ${}_R\mathbf{Mod}$ may not have any left inverse. [3]

   (c) Show that a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is split exact if and only if $B \simeq A \oplus C$. [4]

3. (a) Define product of a family of R-modules $\{M_i\}$. Show that any two products of $\{M_i\}$ are isomorphic. [3]

   (b) Show that $Mor_R(\oplus_{i\in I} N_i, M) \simeq \prod_{i\in I} Mor_R(N_i, M)$. [4]

   (c) Let $f : M \longrightarrow M$ be an R-morphism. If M is of finite height, show that there exists $n \in \mathbb{N}$ such that $M = \ker f^n \oplus Im f^n$. [3]

4. (a) Show that coproduct exists in the category ${}_R\mathbf{Mod}$. [3]

   (b) Prove that an R-module M is a direct sum of n-submodules $N_1, N_2, \cdots, N_n$ if and only if there are n non zero R-morphisms $P_1, P_2, \cdots, P_n : M \longrightarrow M$ such that
   (i) $\sum P_i = Id_M$,
   (ii) $P_i \circ P_j = 0$ for all $i \neq j$. [4]

   (c) Let $f : M \longrightarrow M$ be an R-morphism. Show that $M = Imf \oplus \ker f$ if and only if $Imf = Imf^2$. [3]

5. (a) Let (F, f) be a free R-module on a nonempty set X. Show that f is one-to-one and Imf generates F. [3]

   (b) Show that an R-module F is free if and only if it has a basis. [4]

   (c) Let F be a free R-module, and $f : F \longrightarrow M_1$ and $g : M_2 \longrightarrow M_1$ be two R-morphisms. If g is an epimorphisms, show that there is an R-morphism $h : F \longrightarrow M_2$ such that $g \circ h = f$. [3]

6. (a) Let R be a commutative unitary ring and I be an ideal. Show that a linearly independent subset of the R-module I can not contain more than one element. Hence or otherwise show that if every ideal of R is a free R-module, then R is a PID.. [3]

   (b) Show that direct sum of any family of free R-modules is free. [4]

   (c) Show that the $\mathbb{Z}$-module $\mathbb{Q}_p/\mathbb{Z}$ is not free. [3]

M. A./M. Sc. Examination-2023

Semester-IV

Mathematics

Optional Course : MMO-41 (P03)

(Advanced Real Analysis-II)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer question no.6 and any three from the rest

1. (a) When is a sequence of sets $\{E_n\}$ called convergent? Show that a monotone sequence of sets is convergent.
   (b) Show that a sequence of sets $\{E_n\}$ is monotone if and only if the sequence of its characteristic function $\{\chi_{E_n}\}$ is monotone of the same type. [4]
   (c) Suppose ψ is a completely additive set function defined on an algebra $\mathcal{A}$ and if $\{E_n\} \subset \mathcal{A}$ is an increasing sequence of sets such that $\cup E_n \in \mathcal{A}$, then show that [4]
$$\lim_{n\to\infty} \psi(E_n) = \psi(\lim_{n\to\infty} E_n).$$

2. (a) State and prove Caratheodary Hahn extension theorem [4]
   (b) Define finite and σ-finite measure. Show that the Lebesgue measure on $\mathbb{R}$ is σ-finite. [7]
   (c) Show that the Lebesgue measure is complete. [3]
3. (a) Describe Method-I construction of outer measure μ^*. Show that this outer measure is regular. [2]
   (b) Prove that there exist outer measures which are not regular. [3+2]
   (c) Define metric outer measure. Suppose μ^* is a metric outer measure on a metric space X. Show that all Borel sets in X are μ^*-measurable. [3]
4. (a) Define Lebesgue Stieltjes outer measure. Show that for any semiclosed interval $(a, b]$, $\mu_f^*((a, b]) = f(b) - f(a)$. [1+3]
   (b) Prove that every Borel measure μ is the Lebesgue - Stieltjes measure induced by the distribution function corresponding to μ [2+5]
5. (a) If f and g are integrable on E and $f \geq g$ on E then show that $\int_E f d\mu \geq \int_E g d\mu$ [5]
   (b) State and prove Monotone convergence theorem for a measure space X. Show that for any sequence of non negative measurable functions $\{f_n\}$ defined on X [3]
$$\int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \left(\int_X f_n d\mu\right)$$
[2+2]
   (c) If f is a nonnegative measurable function on X and $\nu(E) = \int_E f d\mu$, then show that ν is a measure on X. Also show that for another nonnegative function g,
$$\int_E g d\nu = \int_E g f d\mu$$

6. Answer any two [2+3]
   (a) Give examples of a complete and an incomplete measure.
   (b) Let X be a non void set. Define $\mu^*(\emptyset) = 0$ and $\mu^*(E) = 1$ for every set $E \subset X, E \neq \emptyset$. Show that μ^* is an outer measure and if X contains more than one point then μ^* is not a measure. [2]
   (c) Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ be a measure space, where μ is counting measure. Show that $\{x_n\}$ is integrable if and only if $\sum x_n$ absolutely convergent. [2]

[2]

M. Sc. Examination-2023

Semester-IV

Mathematics

Course: MMO-41(P-2)

(Advanced Functional Analysis)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any <u>four</u> questions

1. (a) Prove that every closed convex subset of a Hilbert space H has a unique member of smallest norm. [4]

   (b) Show that a Hilbert space H contains a complete orthonormal sequence if and only if H is separable. [4]

   (c) Let $M \subset H$ where H is a Hilbert space. Prove that $M^{\perp}$ (orthogonal complement of M) is a closed subspace of H. [2]

2. (a) Let f be a sesquilinear form defined on a Hilbert space and $\hat{f}$ be the associated quadratic form of f. Prove that f is bounded iff $\hat{f}$ is bounded and moreover $||\hat{f}|| \leq ||f|| \leq 2||\hat{f}||$. [3]

   (b) Prove that for every bounded linear operator $T : H \to H$ where H is a Hilbert space, its adjoint operator T^* exists which is unique and satisfies $||T^*|| = ||T||$. [4]

   (c) Let T be a normal operator defined on a Hilbert space H. Show that $Tx = \lambda x$ iff $T^*x = \overline{\lambda} x \ \ \forall x \in H$ and for all scalars λ. [3]

3. (a) Define normal operator on a Hilbert space. If N_1 and N_2 are normal operators defined on a Hilbert space H such that either commutes with the adjoint of the other. Prove that $N_1 + N_2$ and $N_1 N_2$ are normal operators on H. [4]

   (b) Show that the collection of all normal operators is a closed subset of the set of all bounded linear operators defined on a Hilbert space. [4]

   (c) Give an example of a complete orthonormal sequence in a Hilbert space with justification. [2]

4. (a) Define compact linear operator on a normed linear space and give an example for the same. [3]

(b) Let T be a bounded linear operator defined on a Hilbert space into itself. Show that T^*T is idempotent, self-adjoint and positive. [3]

(c) Show that the spectrum $\alpha(T)$ of a bounded linear operator on a complex Banach space is a closed set (assume $\alpha(T) \neq \phi$). [4]

5. (a) Let $T : H \to H$ be a self-adjoint operator where H is a Hilbert space. Prove that $||T|| = Sup \, \{| < Tx, x > | : ||x|| = 1\}$. [4]

(b) Show that every unitary operator on a Hilbert space is isometric but not conversely. [3]

(c) Prove that the residual spectrum $\alpha_r(T)$ of a bounded self-adjoint linear operator $T : H \to H$ where H is a complex Hilbert space is empty. [3]

6. (a) Define a Banach algebra and give an example for the same. [3]

(b) Let A be a complex Banach algebra with identity e. If $x \in A$ satisfies $||x|| < 1$, then prove that $(e - x)$ is invertible and $(e - x)^{-1} = e + \sum_{i=1}^{\infty} x^i$. [4]

(c) Let $T : H \to H$ be a self-adjoint operator where H is a complex Hilbert space. Show that the eigen vectors corresponding to different eigen values of T are orthogonal. [3]

M.Sc. Examination 2023

Semester-IV

Mathematics

Paper: MMO 41 (P1)

(Advanced Complex Analysis-II)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) State and prove Mittag-Leffler's Theorem. [1+3]
   (b) Let $f(z)$ be regular in $|z| < R$ and $f(0) \neq 0$. If it has zeros at $a_1, a_2, ...a_m$ with moduli not exceeding $r(0 < r < R)$ then show that
   $log\{|\frac{r^m}{a_1, a_2, ..., a_m} f(0)|\} = \frac{1}{2\pi}\int_0^{2\pi} log|f(re^{i\theta})|d\theta$. [3]
   (c) State Poisson-Jensen formula. Prove the result for the function $f(z) = \frac{1}{z-a}, (|a| < R)$ in $|z| < R$. [1+2]

2. (a) Define Nevanlinna's characteristic function $T(r, f)$. Find $T(r, f)$ when $f(z) = e^{z^2}$. [1+2]
   (b) Show that for any a, $|T(r, f + a) - T(r, f)| \leq log^+|a| + log2$. [2]
   (c) State Cartan's identity. Using this prove that $\frac{1}{2\pi}\int_0^{2\pi} m(r, e^{i\theta})d\theta \leq log2$. [1+2]
   (d) Prove that $T(r, f)$ is a convex function of $logr$. [2]

3. (a) Prove that $T(R, f) = T(R, \frac{1}{f}) + log|f(0)|$, $f(0) \neq 0, \infty$. [2]
   (b) State and prove Nevanlinna's first fundamental theorem. [1+3]
   (c) Find the order of the meromorphic function $f(z) = \frac{z+1}{z^2-4}$. [2]
   (d) If ρ_1 and ρ_2 are the orders of the meromorphic functions f_1 and f_2 respectively, then show that the order of $f_1 \pm f_2 \leq max\{\rho_1, \rho_2\}$. [2]

4. (a) Define $\delta(a, f)$ and find $\delta(a, f)$ when $f(z) = 2e^z$ and $a = 0$. [1+2]
   (b) State and prove Nevanlinna's theorem on deficient values. [1+4]
   (c) Show that there can have at most two distinct values a for which the equation $f(z) = a$ has no roots. [2]

5. (a) When two meromorphic functions $f(z)$ and $g(z)$ share a value $'a'$ CM and IM ? [2]
   (b) Let f and g be two non-constant polynomials and $'a'$ be a finite complex number. If f and g share $'a'$ CM then show that there exists a non-zero constant k such that $(f - a) \equiv k(g - a)$. [2]
   (c) Let f and g be two non-constant rational functions. If f and g share two distinct values $'a'$ and $'b'$ CM, then show that $\frac{f-a}{f-b} \equiv k\frac{g-a}{g-b}, k \neq 0$. [4]
   (d) Give an example with proper justification to show that there exist functions f and g share four values IM but $f \neq g$. [2]

6. (a) State and prove Milloux theorem. [1+5]
   (b) If $f(z)$ is a non-constant meromorphic function then prove that $\rho_{f'} \leq \rho_f$. [2]
   (c) Prove that derivative of an elliptic function is elliptic. [2]

M. Sc. Examination-2023

Semester-IV

Mathematics

MMO-41 (P5) : Algebraic coding theory-II

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Suppose C is an $[n, k, d]$-linear code over F_q. Then show that there exists an $[n, k, d-r]$-linear code over F_q for any $1 \leq r \leq d-1$. [5]

   (b) Construct a $[7, 3, 2]$-binary linear code using direct sum of two linear codes. [5]

2. (a) For $m \geq 1$, show that the Reed-Muller code $\mathfrak{R}(1, m)$ is a binary $[2^m, m+1, 2^{m-1}]$-linear code. [5]

   (b) Let C be an $[N, K, D]$-linear code over F_{q^m}. Then prove that the trace code of C defined by

$$Tr_{F_{q^m}/F_q}(C) = \{(Tr_{F_{q^m}/F_q}(c_1), \ldots\ldots, Tr_{F_{q^m}/F_q}(c_n)) : (c_1, \ldots\ldots, c_n) \in C\}$$

   is an $[n, k]$-linear code over F_q with $n = N$ and $k \leq mK$. [5]

3. (a) Let I be a nonzero ideal in $F_q[x]/ < x^n - 1 >$ and let $g(x)$ be a nonzero monic polynomial of the least degree in I. Then $g(x)$ is a generator of I and divides $x^n - 1$. [5]

   (b) How many binary cyclic codes of length 8? Justify your answer. [5]

4. (a) Show that a q-ary BCH code of length q^m with designed distance δ has dimension at least $q^m - 1 - m(\delta - 1)$. [6]

   (b) Find the dimension of a narrow-sense 4-ary BCH code of length 63 with designed distance 3. [4]

5. (a) Show that a BCH code with designed distance δ has minimum distance at least δ. [7]

   (b) Let α be a root of $1 + x + x^3 \in F_2[x]$, and let C be the binary BCH code of length 7 with designed distance 4 generated by $g(x) = lcm(M^0(x), M^1(x), M^2(x))$. Find $d(C)$. [3]

6. (a) Let C be a q-ary RS code generated by $g(x) = \prod_{i=1}^{\delta-1}(x - \alpha^i)$ with $2 \leq \delta \leq q-1$. Then show that the extended code C is still MDS. [6]

   (b) Find a binary [21, 3, 7]- linear code spanned by $100100\ldots100, 010010\ldots010, 001001\ldots001$. [4]