M. Sc. Examination-2023

Semester-I

Mathematics

Paper: MMC-11

(Real Analysis)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Show that the outer measure of an interval is its length.
(b) Define Lebesgue measurable sets. Prove that the collection of all measurable sets in $\mathbb{R}$ is a σ-algebra. [5+(1+4)]

2. (a) Let A be a measurable subset of $\mathbb{R}$. Show that for each $\delta > 0$, there exists an open set B containing A such that $m^*(B \setminus A) < \delta$.
(b) What do you mean by the Lebesgue measure $m(A)$ of a set $A \subset \mathbb{R}$? Explain with examples. Show that m is countably additive. [4+(2+4)]

3. (a) When is a function $f : A \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ said to be (Lebesgue) measurable? Examine if a monotone function defined on an interval is measurable.
(b) Prove or disprove: There exist non-measurable subsets in $\mathbb{R}$. [(1+4)+5]

4. (a) Let $\{f_n\}$ be a sequence of measurable functions on $A \subset \mathbb{R}$ which converges pointwise *a.e.* on A to a function f. Is f measurable? Justify your answer.
(b) Let $f : [0, 8] \longrightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} -1 & \text{if } 0 \le x < 3 \\ 3 & \text{if } 3 \le x < 5 \\ 28 & \text{if } 5 \le x \le 8. \end{cases}$ Compute the Lebesgue integral, if it exists, of f over $[0, 8]$ with proper justification. [5+5]

5. (a) Let $f \equiv \infty$ on an uncountable set A. Can you conclude that $\int_A f = 0$? Justify your answer.
(b) State and prove Monotone Convergence Theorem. Does it hold for decreasing sequences of functions? Provide justification. [4+(3+3)]

6. (a) Describe the notion of convergence in measure and almost uniform convergence. Show that the latter implies the first.
(b) When is a non-negative measurable function f called integrable over a set B? Show that such an f is finite *a.e.* on B. [(2+4)+(1+3)]

M.Sc. Examination - 2023

Semester-I
Mathematics
Paper: MMC 12
(Complex Analysis)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) State and prove Morera's theorem. [1+3]
   (b) State and prove minimum modulus theorem. [1+2]
   (c) State Taylor's theorem. Using this theorem expand $f(z) = \frac{1}{z^2}$ in power series about $z = -1$. [1+2]

2. (a) State Laurent's theorem. Apply Laurent's theorem to show that $e^{z+\frac{1}{z}} = \sum_{n=-\infty}^{\infty} b_n z^n$ valid in $0 < |z| < \infty$, where $\pi b_n = \int_0^\pi e^{2\cos\theta} \cos n\theta d\theta$ and $b_{n-1} - b_{n+1} = nb_n$. [1+3]
   (b) Define zero of an analytic function. If $f(z)$ be a non-constant analytic function regular in a region R, then prove that $f(z)$ have at most a finite number of zeros in every closed sub region of R. [1+2]
   (c) State and prove Schwarz's lemma. [3]

3. (a) State and prove Casorati-Weierstrass theorem. [1+3]
   (b) Show that if $f(z) = \frac{\phi(z)}{(z-\alpha)^n}$, where $\phi(z)$ is analytic at α and $\phi(\alpha) \neq 0$ then α is a pole of $f(z)$ of order n and conversely. [2+2]
   (c) If $z = \alpha$ is a pole of $f(z)$ of order n then show that $\frac{1}{f(z)}$ has a zero of order n at $z = \alpha$. [2]

4. (a) Find the singularities of the function $f(z) = cot(\frac{1}{z})$ and determine the nature of the singularities. [3]
   (b) Establish the formula
   $Z - P = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$
   where the symbols used have their usual meanings. [4]
   (c) Prove that limit point of zeros of a non-constant analytic function is an essential singularity. [3]

5. (a) Find the singularities of the function $f(z) = sin\frac{1}{1-z} + sin\frac{1}{1+z}$. [3]
   (b) State Cauchy's residues theorem. Using this theorem evaluate $\int_C \frac{e^{2z}}{(z+2)^3} dz$, where C is the circle $|z-1| = 2$. [1+2]
   (c) State Rouche's theorem. Show that the roots of the eqnation $16z^5 - z + 8 = 0$ all lie in the annulus between $|z| = \frac{1}{2}$ and $|z| = 1$. [1+3]

6. (a) Define residue of a function at the point at infinity. Find $Res[f(z), \infty]$, where $f(z) = \frac{z}{z^2+1}$. [1+2]
   (b) If $z = \infty$ is a pole of order m for an analytic function $f(z)$ then prove that $f(z)$ is a polynomial of degree m. [2]
   (c) Evaluate any one of the following by the method of contour integration:
   (i) $\int_0^\infty \frac{dx}{x^2a^2+1}, a < 0$; (ii) $\int_{-\infty}^{\infty} \frac{x sin nx dx}{x^2+b^2}, b > 0$. [5]

M. Sc. Examination-2023

Semester-I

Mathematics

MMC-13(Linear Algebra)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
(V is a finite dimensional vector space over F.)

Answer ***any four*** questions.

1. (a) Let $A \in M_2(\mathbb{R})$ be an orthogonal matrix with $\det A = -1$. Find the eigen values of A. Also find the general form of such matrices A. [2+2]
   (b) Let T be a linear operator on V with an eigen value λ. Show that the geometric multiplicity of λ can not exceed its algebraic multiplicity. [3]
   (c) State and prove the Cayley-Hamilton theorem for linear operators. [3]

2. (a) Let T be a linear operator on V. Show that there is the unique monic polynomial $m(t) \in F[t]$ of least degree such that $m(T) = T_0$. [3]
   (b) Give an example of two non similar matrices which have the same minimal and characteristic polynomials. [2]
   (c) Let $A \in M_n(F)$ be a nilpotent matrix. Then show that $A^n = O$. If index of nilpotence of A is n, then show that there can not be a matrix $B \in M_n(F)$ such that $B^2 = A$. [2]
   (d) Let $A, B \in M_n(F)$ and $f(t) = \det(tI_n - B)$. Show that $f(A)$ is invertible if and only if A and B have no common eigen values. [3]

3. (a) Show that a linear operator T is diagonalizable if and only if the minimal polynomial of T is a product of distinct linear factors. [4]
   (b) Let $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Show that J is diagonalizable over $\mathbb{R}$. Find a basis of $\mathbb{R}^3$ of eigen vectors of J. [3]
   (c) Let $M \in M_n(\mathbb{R})$ and $T : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ be defined by $T(A) = MA$. If M is diagonalizable then show that T is diagonalizable. Does the converse hold. [3]

4. (a) Show that a linear operator $T : V \longrightarrow V$ is triangulizable if and only if the characteristic polynomial of T splits into linear factors. [4]
   (b) Let λ be an eigen value of an invertible linear operator T. Show that $G_\lambda(T) = G_{\lambda^{-1}}(T^{-1})$. [3]
   (c) Let A be a 4×4 nilpotent matrix. Find all possible Jordan canonical forms of A, and the index of nilpotence and geometric multiplicity of 0 corresponding to each form. [3]

5. (a) State and prove the Riesz representation theorem. [3]
   (b) Let T be a linear operator on $\mathbb{R}^2$ that represents the counterclockwise rotation at an angle θ. Show that $T_\theta^* = T_{-\theta}$. [2]
   (c) If $f : P_2(\mathbb{R}) \longrightarrow \mathbb{R}$ is defined by $f(p) = p(1) + p'(2)$, find a vector $q \in P_2(\mathbb{R})$ such that $f(p) = < p, q >$ for all $p \in P_2(\mathbb{R})$, where $< p, q > = \int_0^1 p(x)q(x)dx$. [3]
   (d) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear operator. Show that $\ker T^* = (Im\ T)^\perp$. [2]

6. (a) Let U be a complex inner product space and $T : U \longrightarrow U$ be a linear operator. Prove that T is a normal operator if and only if it is unitarily diagonalizable. [4]
   (b) Let V be an inner product space and $P : V \longrightarrow V$ be an idempotent linear operator. Show that P is self adjoint if and only if $PP^* = P^*P$. [3]
   (c) Prove that a linear operator T on an inner product space V is unitary if and only if $\|T(v)\| = \|v\|$ for all $v \in V$. [3]

M.Sc. Examination-2023

Semester-I

Mathematics

MMC-14

(Ordinary Differential Equations)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions out of six.

1. (a) For the linear autonomous system

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy,$$

where a,b,c, and d are real constants.
(i) Show taht if $a = d$, and b and c are of opposite sign, then the critical point $(0,0)$ of the given system is a spiral point.
(ii) Show that if $a = d$, and either b or c=0, then the critical point $(0,0)$ of the given system is a node.
(iii) Show that if $a = -d$ and $ad > bc$, then the critical point $(0,0)$ of the given system is a center.
(iv) Show that if $a = -d$ and b and c are of the same sign, then the critical point $(0,0)$ of the given system is a saddle point. [4]

(b) Find the maximal interval of existence for the unique solution of the following systems. [3]
$x'(t) = x^2 + cos^2(t)$, $t > 0, x(0) = 0$.

(c) Apply Picard Iteration method to find first three approximation of the following IVP.
$y' = x + y^2$, $y(0) = 0$. [3]
Using Induction show that the solution tends to a unique solution.

2. (a) State and Prove Sturm's Seperation Theorem. [5]

(b) Solve the following BVP

$$\frac{d^2y}{dx^2} = f(x), \qquad y(0) = y(l) = 0; \qquad 0 < x < l,$$

by constructing the Green's function. [5]

3. (a) Show that the coefficient matrix A can be expressed in terms of the Fundamental Matrix $\Phi(t)$ of the system $\dot{x} = Ax$. [5]

(b) Consider a second-order homogeneous linear differential equation and write its adjoint equation. When is a differential equation called self-adjoint? Find the necessary and sufficient condition that a second-order linear differential equation $a_0(t)\frac{d^2x}{dt^2} + a_1(t)\frac{dx}{dt} + a_2(t)x = 0$ be self-adjoint. [1+1+3]

4. (a) If N be a nilpotent matrix of order k. Then show that e^{Nt} is a series containing finite terms only. Under which condition we can write $e^{A+B} = e^A e^B$? [1+3+1]

(b) Prove that for a Strum-Liouville problem the distinct eigen functions are orthogonal corresponding to distinct eigen values. [5]

5. (a) Define a Sturm-Liouville Problem. Under which condition a Sturm-Liouville problem is called Regular and Singular?
Find the eigen-values and the eigen-functions of the Sturm-Liouville problem $\frac{d^2y}{dx^2} + \lambda y = 0$, with $y(0) + y'(0) = 0$, $y(1) + y'(1) = 0$. [1+1+4]

(b) With a suitable example show that Lipchitz criteria is a weaker concept than Picard's condition for uniqueness of solutions of a first order IVP. [2]

(c) Find the Natural Fundamental set of solutions of the differential equation $y'' + 4y = 0$, with the initial time 0. [2]

6. (a) Find the solution of the non-homogeneous system
$\dot{x} = x + y + t,$
$\dot{y} = -y + 1$
with the initial conditions $x(0) = 1, y(0) = 0$, using Fundamental Matrix. [5]

(b) State Peano's Theorem for existence of first order IVP. Is the condition necessary ? Justify your answer with a suitable example. [3]

(c) Find the 'interval of definition' for the solution of the following IVP. [2]
$\frac{d^4y}{dt^4} + \frac{1}{t-4}\frac{dy}{dt} = \frac{e^t}{2+t}, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0.$

M. Sc. Examination-2023
Semester-I
Mathematics
Paper: MMC 15
(Partial Differential Equations)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notation and symbol have their usual meanings.

Answer ***any four*** questions.

1. (a) Consider the Cauchy problem

$$(x+2)\,u_x + 2y\,u_y = 2u$$

in xyu - space and $u(-1,y) = \sqrt{|y|}$. Let $u(x,y)$ be the solution to the Cauchy problem, find the value of $u(-2,1)$. Is the solution unique? Justify your answer.

(b) Show that the complete integral of the partial differential equation (PDE)

$$u = x\,u_x + y\,u_y - 4\,u_x - 5\,u_y$$

represents all possible planes passing through the point $(4,5,0)$. Also, determine the envelope of all planes represented by the complete integral.

(c) Define Monge cone of a first order nonlinear PDE $f(x,y,u,u_x,u_y) = 0$ at any point (x_0,y_0,u_0). Find the expression of the Monge cone of the following PDE at $(1,2,3)$:

$$x^2\,u_x^2 + y^2\,u_y^2 = u^2.$$

[(3+1)+3+(1+2)]

2. (a) Find the complete integral of the PDE

$$u_{xx} - u_{xy} - 2u_x = e^{3x+y} + \cos(3x+y).$$

(b) Solve the following PDE by reducing it to the canonical form:

$$x^2\,u_{xx} - 2xy\,u_{xy} + y^2\,u_{yy} - x\,u_x + 3y\,u_y = \frac{8y}{x}.$$

[4+6]

3. (a) Find the number of real characteristic curves of the PDE

$$\left[\alpha(x,y)\frac{\partial}{\partial x} + \beta(x,y)\frac{\partial}{\partial y}\right]\left[\gamma(x,y)\frac{\partial}{\partial x} + \delta(x,y)\frac{\partial}{\partial y}\right]u = 0,\ \alpha\delta - \beta\gamma = 0,$$

where α, β, γ, δ be four differentiable functions defined on $\mathbb{R}^2$.

(b) With the help of Cauchy's method of characteristics, find the integral surface of the PDE

$$xu_x + yu_y = u_x u_y$$

which passes through the curve $u = x^2$, $y = 0$.

(c) Solve the following PDE by Monge's method:

$$u_{xx} - 2024\,u_{xy} + (1012)^2\,u_{yy} = 0.$$

[1+4+5]

4. (a) Using appropriate integral transform method, solve the initial boundary value problem (IBVP) described as

$$\epsilon^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1,\ 0 < t < \infty,$$

BC's: $u(0,t) = 1,\ u(1,t) = e^{-2t},\ 0 < t < \infty,$
IC: $u(x,0) = 1 + \sin(\pi x),\ 0 \leq x \leq 1.$

(b) Using Fourier integral transform method, solve the following two-dimensional (2D) Laplace equation:

$$u_{xx} + u_{yy} = 0,\ 0 < x < \pi,\ 0 < y < \pi,$$

BC's: $u(x,0) = 0,\ u(x,\pi) = u_0,$
$u(0,y) = 0 = u(\pi,y)$ for every y and $u(x,y)$ is bounded. [5+5]

5. (a) Show that the d'Alembert's solution of free vibrations of a semi-infinite string governed by

$$\begin{aligned} u_{tt} &= c^2 u_{xx},\ 0 < x < \infty,\ t > 0, \\ \text{subject to: } u(x,0) &= \phi(x), \\ u_t(x,0) &= \psi(x), \end{aligned}$$

is $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)d\xi.$

(b) Consider the following heat conduction problem for a finite rod

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - xe^t - 2t,\ x \in (0,\pi),\ t > 0,$$

BC's: $u(0,t) = -t^2,\ u(\pi,t) = -\pi e^t - t^2,\ t > 0,$
IC: $u(x,0) = \sin x - \sin^3 x - x,\ 0 \leq x \leq \pi.$
If $v(x,t) = u(x,t) + xe^t + t^2$, find the complete expression of $v(x,t)$.

(c) By using the method of separation of variables, find the value of $u(3,3)$ of the following PDE:

$$x^2 u_{xy} + 3y^2 u = 0,\ u(x,0) = e^{\frac{3}{x}}.$$

[4+3+3]

6. (a) Prove that the solutions of the Dirichlet problem for the Laplace equation in some function domain $\mathbb{R}$ and in some boundary of the domain $\partial\mathbb{R}$ depend continuously on the boundary data.

(b) Solve the following interior Neumann problem for a circle:
PDE: $\nabla^2 u = 0,\ 0 \leq r < a,\ 0 \leq \theta < 2\pi,$
BC: $\frac{\partial u}{\partial r} = g(\theta)$ at $r = a,\ 0 \leq \theta < 2\pi,$
where $g(\theta)$ is a continuous function on $\partial\mathbb{R}$. [3+7]

M. Sc. Examination-2023

Semester-I
Mathematics
Paper: MMC-16
(Integral Transforms and Integral Equations)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer Question No. 1 and any *three* from the rest.

1. Answer any *five* questions. [5 × 2 = 10]

 (a) Using modified Adomian decomposition method, solve the integral equation
 $y(x) = 1 - x + 3x^2 + cosx - x^3 - sinx + \int_0^x y(t)dt.$ [2]

 (b) Solve the Fredholm integro-differential equation
 $u'(x) = 3 + 4x - x^2 + \int_0^2 u(t)dt,\ u(0) = 0.$ [2]

 (c) Find the Laplace transform of $f(t)$, where $f(t) = sint$ when $0 < t < \pi$ and $f(t) = sint + cost$ when $t > \pi$. [2]

 (d) Find the eigenvalue and eigenfunction of the homogeneous integral equation
 $y(x) = \lambda \int_0^2 x^2t^5\, y(t)dt.$ [2]

 (e) Find the inverse Laplace transform of $F(s)$, where $F(s) = \frac{1}{s^2+5s+6} + \frac{1}{s(s-7)^3}$. [2]

 (f) Convert the integral equation $y(x) = x^3 - x + \int_0^1 k(x,t)y(t)dt$ to an equivalent differential equation, where $k(x,t) = 9t$ when $0 \le t \le x$ and $k(x,t) = 9x$ when $x \le t \le 1$. [2]

 (g) Find the value of $f(4)$, where $f(t) = e^{-3t} * e^{-t} * e^{-7t}$. [2]

2. (a) Using Hilbert-Schmidt theorem, solve the integral equation
 $y(x) = 1 + \int_0^1 (1 - 3xt)y(t)dt.$ [4]

 (b) Convert the differential equation $\frac{d^2y}{dx^2} - sinx\frac{dy}{dx} + e^x y = x, y(0) = 1, y'(0) = -1$ to an equivalent Volterra integral equation. [4]

 (c) Using the complex inversion formula, find the inverse Laplace transform of the function $F(s) = \frac{s}{s^2+9}$. [2]

3. (a) If f and g are piecewise continuous on $[0,\infty)$ and of exponential order, then prove that $L\{f * g\} = L\{f\}.L\{g\}$. [2]

 (b) Find the resolvent kernel of the integral equation $y(x) = x^7e^x + \lambda \int_{-1}^1 (xt + x^5t^5)y(t)dt.$ [4]

 (c) Using Laplace transform, solve the differential equation
 $\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2e^t, y(0) = 1, y'(0) = 0, y''(0) = -2.$ [4]

4. (a) State Fredholm alternative theorem. Find the conditions for which the integral equation $y(x) = f(x) + \lambda \int_0^{\frac{\pi}{2}} cos(x - t)y(t)dt$ has infinitely many solutions. [1+4]

(b) Show that Volterra integral equation of first kind can be converted to a Volterra integral equations of second kind.
Hence solve the Volterra integral equation $\int_0^x (2 + x^2 - t^2)y(t)dt = x^2$. [2.5+2.5]

5. Solve the Volterra integral equation $y(x) = x - \int_0^x (x - t)y(t)dt$ by

(a) Laplace transform method, [2]

(b) series solution method, [2]

(c) Adomian decomposition method, [2]

(d) successive approximations method, [2]

(e) successive substitutions method. [2]

6. (a) Find the Fourier transform of $f(x)$,
where $f(x) = 1$ if $|x| < a$ and $f(x) = 0$ if $|x| > a$.
Hence using Persaval's identity, evaluate $\int_0^{\infty} (\frac{sinax}{x})^2 dx$. [2+2]

(b) Find the Fourier sine transform of $\frac{e^{-ax}}{x}$. [3]

(c) Find the Fourier cosine transform of $f(x)$, where $f(x) = cos2x$ when $0 < x < 7$, and $f(x) = 0$ when $x > 7$. [2]

(d) Solve the differential equation $\frac{dy}{dt} + 5y = \delta(t)$. [1]