Use separate answer script for each unit

B. Sc. (Honours) Examination-2023

Semester-VI (CBCS)
Mathematics
Course: CCMA-13
(Analysis-VI)

Time: 3 Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 30)
(Complex Analysis)

Answer ***any three*** questions.

1. (a) If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a function such that $\lim_{z \to z_0} f(z) = \overline{w}$, where $w \in \mathbb{C}$, then show that $\lim_{z \to z_0} \overline{f(z)} = w$. [3]
   (b) If a continuous function $f : D \longrightarrow \mathbb{C}$, defined on a domain D, takes only the real integral values, show that f is constant. [4]
   (c) Derive Cauchy-Riemann equations in complex form. [3]
2. (a) Examine if $u(x, y) = y + e^x \cos y$ is a harmonic function in $\mathbb{C}$. If possible, find its harmonic conjugate and then construct an analytic function with $u(x, y)$ as its real part. [4]
   (b) Discuss the convergence of the series $\sum_{1}^{\infty} n(1 + i)(2i)^{-n}$. [3]
   (c) Let $f : \mathbb{C} \longrightarrow L$ be analytic, where L is the line $x + 2y = 1$. Prove or disprove: f is bounded in $\mathbb{C}$. [3]
3. (a) Find all the Möbius transformations of the unit disc $|z| \leq 1$ onto the right half-plane $Re(w) \geq 0$. [5]
   (b) Evaluate $\int_\Gamma (\overline{z}dz + zd\overline{z})$ along the curve Γ defined by $z^2 - 2z\overline{z} + (\overline{z})^2 = (1 + i)z + (1 - i)\overline{z}$ from the point $z = -1 + i$ to the point $z = -6 + 2i$. [5]
4. (a) State and prove Cauchy's integral formula. [4]
   (b) Describe the orientation principle related to Möbius transformations. Hence find the image of the exterior of the circle $|z + 2| = 2$ under the map $w(z) = \dfrac{z}{2z + 8}$. [1+3]
   (c) What do you mean by the cross ratio of four complex numbers. Explain how it is related to Möbius transformations. [1+1]
5. (a) Using the techniques of Complex Analysis, establish the fundamental theorem of Algebra. [4]
   (b) If f is an entire function satisfying $f(z) = f(z + z_1) = f(z + z_2)$, $\forall z \in \mathbb{C}$, where z_1 and z_2 are two non-zero complex numbers such that $\frac{z_1}{z_2} \notin \mathbb{R}$, show that f is constant. [4]
   (c) Evaluate $\int_\Gamma \overline{z}dz$, where Γ is the right-half of the circle $|z| = 1$ from $z = -i$ to $z = i$. [2]

Unit-II (Full Marks: 30)
(Metric Spaces)

Answer ***any three*** questions.

6. (a) Define a metric space. Give an example of it with proper justifications. [2+1+2]
   (b) Let X be a non empty set and ρ be a real valued function of ordered pairs of elements of X which satisfies the following two conditions:
   $\rho(x, y) = 0$ if and only if $x = y$ and $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$. Show that $\rho(x, y)$ is a metric on X. [3]
   (c) Let R be the set of all real numbers and for x, y in R, let $\rho(x, y) = |x^4 - y^4|$. Show that (X, ρ) is not a metric space. [2]

7. (a) Define a convergent sequence in a metric space (X, ρ). Show that in a metric space (X, ρ), if $x_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$ then $\rho(x_n, y_n) \to \rho(x_0, y_0)$. [1+2]

(b) Show that necessary and sufficient condition that a sequence $\{x_n\}$ of points of X converges to $x \in X$ is that every neighbourhood $S(x, a)$ of x contains all the points of the sequence except perhaps a finitely many. [2+2]

(c) Is the arbitrary intersection of open sets open in a metric space? Answer with reason. [1+2]

8. (a) Define the closure $\bar{A}$ of a set $A \subset X$ in a metric space (X, ρ). Prove that a point $p \in \bar{A}$ if and only if $S \cap A \neq \emptyset$ for every neighbourhood S of p. [1+2+2]

(b) If A, B are subsets of X in a metric space (X, ρ), then show that
(i) $A \subset \bar{A}$, (ii) $A \subset B$ implies $\bar{A} \subset \bar{B}$, (iii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$. [1+2+2]

9. (a) Give an example with proper justification of a Cauchy sequence in a metric space. [2]

(b) Prove that a Cauchy sequence, which has a convergent subsequence is itself convergent. [3]

(c) Define a complete metric space. Prove that $C[a, b]$ is a complete metric space. [1+4]

10. (a) Let X be a metric space and Y be a subset of X. Then prove that Y, considered as a metric space is closed, if it is complete. [1+3]

(b) State and prove Banach's contraction mapping principle. [6]

B. Sc. (Hons) Examination-2023

Semester-VI

Mathematics

CCMA 14 (Algebra IV)

Time: Three Hours

Full Marks: 60

Questions are of values as indicated in the margin.
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Group - A (Group Theory)

Answer ***any four*** questions.

1. (a) Let $G = \{\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \mid r \in \mathbb{R}\}$. Show that the group G is isomorphic to $\mathbb{R}$. Are the groups $\mathbb{R}^*$ and $\mathbb{R}^+$ isomorphic? [2+1]

   (b) Consider the group homomorphism $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ defined by $f(z) = z^{100}$. Find the kernel and image of f. [2]

2. (a) Let $f : G \longrightarrow G'$ be a homomorphism and H be a subgroup of G. Show that $f^{-1}(f(H)) = H \ker f$. [2]

   (b) Prove that the group $\mathbb{Q}/\mathbb{Z} \simeq G$ where $G = \cup_{n\in\mathbb{N}} C_n$ is the group of all roots of unity. [3]

3. (a) State and prove the second isomorphism theorem for groups. [3]

   (b) Define a suitable homomorphism $f : GL_2(\mathbb{R}) \longrightarrow \mathbb{R}^*$ to show that the order of the group $GL_2(\mathbb{R})/N$ is two where $N = \{A \in GL_2(\mathbb{R}) \mid \det A > 0\}$. [2]

4. (a) Find all homomorphisms from $\mathbb{Z}$ onto $\mathbb{Z}_9$. [2]

   (b) Is $\mathbb{Z}$ a homomorphic image of $\mathbb{R}$? Justify. [3]

5. (a) Show that $GL_2(\mathbb{R})$ is not isomorphic to $\mathbb{R}^*$. [2]

   (b) Show that an infinite group G is cyclic if and only if it is isomorphic to each of its nontrivial subgroups. [3]

6. (a) Find $Aut(K_4)$. [2]

   (b) Prove that $Aut(\mathbb{Z}_n) \simeq U_n$. [3]

Group - B (Ring Theory)

Answer ***any four*** questions.

1. (a) State and prove the correspondence theorem for rings. [3]

   (b) Let R be a ring with 1. If the characteristic of R is a prime p, then show that R has a subring isomorphic to $\mathbb{Z}_p$. [2]

2. (a) Find all ring homomorphisms $f : \mathbb{Z} \longrightarrow \mathbb{Z}$. [2]

   (b) Let X be a nonempty set and $R = Map(X, \mathbb{Z}_2)$ be the ring of all mappings from X into $\mathbb{Z}_2$. For every $a \in X$ define $C_a : X \longrightarrow \mathbb{Z}_2$ by

$$C_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

   Show that as rings $\mathcal{P}(X) \simeq R$. [3]

3. (a) Show that no two subrings of $\mathbb{Z}$ are isomorphic. [2]

   (b) Let R be a ring homomorphic image of $\mathbb{Z}$. Show that R is a commutative ring with unity and every ideal of R is principal. [3]

4. (a) Use correspondence theorem to find the number of ideals of $\mathbb{Z}_{100}$. [1]

   (b) Let R be a commutative ring with 1, and I and J be two ideals of R such that $R = I + J$. Show that $IJ = I \cap J$ and $R/IJ \simeq R/I \times R/J$. [4]

5. (a) Find all maximal ideals of $\mathbb{Z}_{100}$. [2]

(b) Let R be a commutative ring with 1 and M be an ideal of R. Show that M is maximal if and only if R/M is a field. [3]

6. (a) Show that the ideal $< x >$ in $\mathbb{Z}[x]$ is a prime ideal. Is $< x >$ a maximal ideal? [3]

(b) Show that every prime ideal in a Boolean ring with 1 is maximal. [2]

Group - C (Linear Algebra)
Answer *any four* questions.

1. (a) A linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by $T(x, y, z) = (x - y + 2z, x + 2y - z, 2x + y + z)$. Find the rank and nullity of T. [2]

(b) Is there any linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $\ker T = \{(x, y, z) \in \mathbb{R}^3 \mid 2x = y = 3z\}$ and $Im\ T = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$? [3]

2. (a) Let A be an $m \times n$ real matrix. Define $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $T(X) = AX$. Show that T is one-to-one if and only if $r(A) = n$. Then show that $ST = TS$ for every $S \in \mathcal{L}(V)$ if and only if $T = cI_V$ for some scalar c. [3]

(b) A linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $T(x, y) = (2x - 3y, x + 4y)$. Find the image of the line $x + 2y = 1$ under T. [2]

3. (a) Let V and W be two finite dimensional vector spaces and U is a subspace of V. Prove that there is a linear transformation $T : V \longrightarrow W$ with $\ker T = U$ if and only if $\dim U \geqslant \dim V - \dim W$. [3]

(b) Let $T : V \longrightarrow W$ be a linear transformation. Show that V has a subspace U such that $V = \ker T \oplus U$ and $ImT = T(U)$. [2]

4. (a) State and prove the rank-nullity theorem. [3]

(b) Let $D, I : P(R) \longrightarrow P(R)$ be the differential and integral operators. Show that $\{D, I\}$ are linearly independent. [2]

5. (a) Let $T : V \longrightarrow V$ be a linear operator such that $T^2 = cT$ for some $c \in F$. Prove that $V = ImT \oplus \ker T$. [2]

(b) Let $S, T : U \longrightarrow V$ be two linear transformations. Then show that $|\text{rank}(S) - \text{rank}(T)| \leqslant \text{rank}(S+T) \leqslant \text{rank}(S) + \text{rank}(T)$. [3]

6. (a) Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ be the matrix of a linear transformation $T : \mathbb{R}^3 \longrightarrow P_2(\mathbb{R})$ in the basis $\beta = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ and $\gamma = \{1 + t, t + t^2, 1 + t^2\}$. Find T. [3]

(b) Show that the rank of a linear transformation is same as the rank of each of its matrix representations. [2]

B.Sc. (Honours) Examination, 2023

Semester-VI (CBCS)

Mathematics

Course: DSEMA-3A

(Computer Fundamentals)

Time: Two Hours | Full Marks: 20

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Write two differences between compiler and assembler. [2]

   (b) A Fibonnaci sequence is defined as follows: $0, 1, 1, 2, 3, 5, \cdots$ $(F_{n+1} = F_n + F_{n-1}, n \geq 1)$. Using standard symbols, draw a flowchart to find all the numbers of this sequence which are ≤ 2023. [3]

2. (a) Why are the following unacceptable as FORTRAN 77 real constants? [2]
   (i) $5E-7$ (ii) $E+8$ (iii) $8, 987.43$ (iv) $7.6E2.8$

   (b) Write a FORMAT statement to print an integer value -2023. [1]

   (c) What do you mean by character expression? Explain with a suitable example. [2]

3. (a) Assume that M and N are integer variables having the values -75 and 78 respectively, and that X, Y, Z are real variables with values -73.56, 80.0 and 114.7 respectively. Find the value of the following expression. [3]
   $.NOT. ((M\ .GT.\ N)\ .AND.\ (X\ .LT.\ Z))\ .NEQV.\ ((M\ .LE.\ N)\ .OR.\ (X\ .GE.\ Z))$

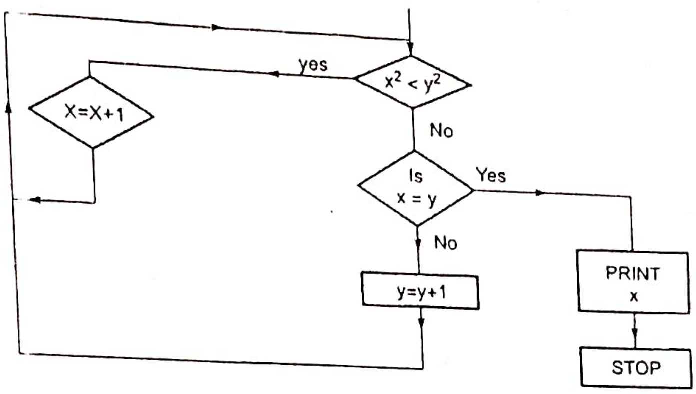
   (b) State two main differences between READ statement and DATA statement. [2]

4. (a) The graph of the equation $x^2 + y^2 = 50$ is a circle with center at the origin and radius $\sqrt{50}$. Write a FORTRAN 77 program to determine the number of points with integer coordinates which lie in the circle. [3]

   (b) Write an arithmetic IF statement equivalent to: [1]
   $GO\ TO(15, 25, 35),\ JACK$

   (c) Point out the errors (if any) in the following: [1]
   $DIMENSION\ M(50)$
   $S = 0.0$
   $DO\ 15\ J = 1,\ 3$
   $15\ IF(M(J)\ .EQ.\ 0)\ S = S + J$

5. (a) Write the FORTRAN 77 program segment for the following flowchart using IF-THEN-ELSE statement. [3]



(b) Rewrite the following FORTRAN 77 segment using a DO loop. [2]

$KOUNT = 0$
$I = I + 1$
$10\ READ(*, 5)\ A$
$IF(A\ .GT.\ 0)\ KOUNT = KOUNT + 1$
$I = I + 1$
$IF(I\ .LE.\ 250)\ GO\ TO\ 10$

6. (a) Find the output (mentioning all the necessary steps involved) of the following program: [3]

$INTEGER\ X(6)$
$DO\ 10\ K = 1,\ 6,\ 2$
$X(K) = 3 * K$
$X(K + 1) = K + 2$
$10\ CONTINUE$
$DO\ 20\ J = 1,\ 6,\ 3$
$X(J) = X(J) + X(J + 1)$
$20\ CONTINUE$
$IF(X(2)\ .LT.\ 7)\ X(2) = X(3)$
$WRITE(6, 15)\ X$
$15\ FORMAT(1X, 6I8)$
$STOP$
END

(b) Define a statement function to calculate $Z = \sqrt{a_1b_1 + a_2b_2}$ and write an assignment statement to calculate $A = \dfrac{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}}{\sqrt{x_1y_1 + x_2y_2}}$. [2]

B. Sc. (Honours) Examination-2023

Semester-VI (CBCS)

Mathematics

Paper: DSEMA-4

(Mathematical Modelling)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Group A (Physical System)

(Full Marks: 30)

Answer ***any five*** questions.

1. How is a simple Pendulum mathematically modeled? Linearized it to convert into a 'simple harmonic motion (SHM)'. If a periodic force of the form $f_0 \cos\omega t$ is added to the SHM, explain the resonance. [2+1+3]

2. Derive the heat equation when there is no loss or gain of heat. Explain each parameter of the equation. Also, write the initial and boundary conditions of the equation. [3+2+1]

3. What is meant by a 'perturbation'? For a damped harmonic motion $\ddot{X} + \epsilon\dot{X} + \omega^2 X = 0$, where ϵ is the perturbation parameter, obtain the solution up to second order perturbation. [2+4]

4. (a) Write the difference between the Static model and the Dynamic model.
   (b) Write the difference between the Explicit model and the Implicit model.
   (c) Write the difference between the Discrete model and the Continuous model.
   (d) Write the difference between the Deterministic model and the Stochastic model.

   [1.5+1.5+1.5+1.5]

5. Explain Over damping, Under damping, and Critical damping for a damped harmonic oscillator. How do you solve a damp harmonic oscillation where the external force is of the form $f_0 \cos pt$. [1+1+1+3]

6. (a) What do you mean by the Physical and Mathematical Models?
   (b) How do they differ from Real World Problems?

   [3+3]

7. With a schematic diagram, design a 'LRC circuit'. Obtain the differential equation of the circuit. Solve the differential equation by explaining each parameter in the solution. [2+1+3]

8. Define a 'SHM'. Obtain the mathematical model of SHM. Convert the equation into a system of an ordinary differential equation $Ax = 0$. Find the eigenvalue of the matrix A and the nature of the fixed point. [1+2+1+2]

Group B (Biological System)
(Full Marks: 30)

Answer question number 1 and *any two* from the rest.

1. Answer *any two* questions.
(a) Define an autonomous system and describe the different kinds of equilibrium points of the system of non-linear ordinary differential equations.
Consider the system

$$\frac{dx}{dt} = x(1-x-y),$$
$$\frac{dy}{dt} = \beta(x-\alpha)y;$$

α, β are positive constants. Show that the non-trivial equilibrium point $(\alpha, 1-\alpha)$ is stable if $\alpha < 1$.
(b) What do you mean by phase plane analysis in dynamical systems? Sketch the phase diagram for the following linear autonomous system and classify the equilibrium point:

$$\frac{dx}{dt} = x + 2y,$$
$$\frac{dy}{dt} = 5x + 4y.$$

(c) In a two-dimensional ($2D$) autonomous system, it is well known that there can be no limit cycles in the model of mutualistic symbiosis populations - Validate the statement via mathematical analysis. [2 × 5=10]

2. Develop a single species age-independent population logistic model. Show that the population growth curve has a point of inflexion when half the final population size is attained. Show also that the point of inflexion occurs at $t = \frac{1}{a} log_e \left[\frac{(\frac{a}{b})}{N_0} - 1\right]$, where a and b are the intrinsic growth rate and intra-specific competition factor respectively and N_0 is the initial population size. [2+4+4]

3. Write the limitations of the Lotka-Volterra predator-prey model. Carry out the stability analysis about the co-existence equilibrium point. Specify the solution trajectory of the model, taking into account the size of each species. [2+4+4]

4. What do you mean by a reaction-diffusion system? Using Fick's law, develop a single-species reaction-diffusion model in one spatial dimension. Include some examples of both linear and non-linear reaction-diffusion models. [2+6+2]