M. Sc. Examination-2024

Semester-II

Mathematics

Course: MMC-21

(Functional Analysis)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any <u>four</u> questions

1. (a) When is a set in a metric space (X, d) called (i) bounded and (ii) totally bounded ? Show that a totally bounded set is bounded but converse is not true. [6]

   (b) Define Lebesgue number. Prove that in a sequentially compact metric space every open cover has a Lebesgue number. [4]

2. (a) When two norms in a linear space are said to be equivalent ? Prove that any two norms defined on a finite dimensional linear space are equivalent. [3]

   (b) Verify that l_∞ (sequence space) is a Banach space w.r.t. a norm to be defined by you. [4]

   (c) Consider the normed linear space $C[0,1]$ w.r.t.supnorm. Let $t_0 \in [0,1]$ be a fixed element and define a function f on $C[0,1]$ by $f(x) = x(t_0)\ \forall x \in C[0,1]$. Prove that f is a bounded linear operator and find $||f||$. [3]

3. (a) Define $||T||$ of a bounded linear operator T over a normed linear space X and show that $||T|| = Sup\{\frac{||Tx||}{||x||} : x(\neq \theta) \in X\}$. [4]

   (b) Prove that every linear operator defined on a normed linear space is continuous iff it is bounded. [3]

   (c) Give an example with justification of a linear operator over a normed linear space which is not bounded. [3]

4. (a) State and prove Hahn-Banach theorem over a normed linear space. [4]

   (b) Show that for each nonzero vector x_0 in a normed linear space $(X, ||\ ||)$, there exists a member $f \in X^*$ such that $f(x_0) = ||x_0||$ with $||f|| = 1$. Hence verify that X^* distinguishes points of X. [4]

   (c) Prove that every open ball in a normed linear space is a convex set. [2]

5. (a) In an inner product space $(X, <\ \ >)$, prove that $| < x, y > | \leq ||x||||y||\ \forall x, y \in X$ where $||x||^2 =< x, x >$ and hence show that X is a normed linear space w.r.t. the norm induced by the inner product. [5]

   (b) Show that $\mathbb{R}^n$ is a Hilbert space w.r.t. an inner product to be defined by you. [3]

   (c) Prove that inner product is a continuous function. [2]

6\. (a) Prove that a Banach space X is a Hilbert space iff Parallelogram law holds in X. [5]

(b) Give an example of an orthonormal set in an inner product space. Show that any orthonormal set of vectors in an inner product space are linearly independent. [3]

(c) Prove that in an inner product space X, any two vectors x, y are orthogonal iff $||x + \alpha y|| = ||x - \alpha y||$ for any scalar α where $|| \ ||$ is the norm induced by the inner product. [2]

M. Sc. Examination-2024

Semester-II

Mathematics

Course : MMC-22 (Topology)

Time: Three Hours   Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any **four** questions

1. (a) Write down Zorn's lemma and Hausdorff maximality principle. Show that Hausdorff maximal principle implies Zorn's lemma. [2+2]
   (b) Show that $c + c = c$ and $a.c = c$. [3]
   (c) Define initial segment of a subset A of a well ordered set X. If α, β, γ are ordinal numbers and $\alpha < \beta$ & $\beta < \gamma$ then show that $\alpha < \gamma$. [3]

2. (a) Give definition of basis for a topological space. Write down a basis for the following topological spaces:
   (*i*) $\mathbb{N}$ with usual subspace topology, (*ii*) $\mathbb{R}$ with discrete topology, (*iii*) $\mathbb{Q}$ with VIP-topology with vip zero. [1+3]
   (b) Write down with justification conditions for which a collection of sets $\mathcal{B}$ would be a basis for some topology. [3]
   (c) What is open map? Give example of an open map which is not continuous. Show that a bijective open map is also a closed map. [3]

3. (a) Show that a T_3 space is a T_2 space but converse is not true. [2+2$\frac{1}{2}$]
   (b) Show that X is T_2 if and only if the set $\Delta = \{(x, x) : x \in X\}$ is a closed set. [3]
   (c) Let X and Y are topological spaces and $f, g : X \to Y$ be continuous functions. If f and g agree on a dense set and Y is T_2, then show that they are identical functions. [2$\frac{1}{2}$]

4. (a) Show that $\mathbb{R}$ with lower limit topology is 1st countable but not 2nd countable . [1+2]
   (b) Let X be 1st countable space , $A \subset X$ and $a \in X$. then show that $a \in \overline{A}$ if and only if there is a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = a$. Also show that "only if" part of above is not true without assuming that the space X is 1st countable. [3+2]
   (c) Show that every subset of a [redacted] topological space is compact. [2]
   co-finite

5. (a) Show that continuous image of a compact set is compact. [2]
   (b) Prove that a space is compact if and only if each family of closed sets with the finite intersection property has a non empty intersection. [3]
   (c) Show that a compact T_2 space is a T_3 space. [3]
   (d) Prove that a metric space is a T_4 topological space. [2]

6. (a) Give definition of connected topological space. Show that the following three statements are equivalent:
   (*i*) X can not be expressed as a union of two disjoint open sets.
   (*ii*) X can not be expressed as a union of two disjoint closed sets.
   (*iii*) The only sets that are both open and closed in X are X and $\emptyset$. [5]
   (b) Show that continuous functions preserve connectedness. [3]
   (c) Show that each function from a compact space to T_2 space is a closed map. [2]

M. Sc. Examination-2024

Semester-II
Mathematics
MMC-23
(Abstract Algebra)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Is $\mathbb{R}^* \times \mathbb{R}^*$ isomorphic to C^*? Justify your answer. (as groups) [2]
   (b) Find the class equation for D_4. [3]
   (c) State and proof Cauchy's Theorem on finite groups. How many elements of order 7 are there in a group of order 28? [1+3+1]

2. (a) Show that the number n_p of Sylow p-subgroups in a finite group of order $p^r m$ is given by $n_p \equiv 1(\text{mod}p\ )$, where p is a prime number and $r, m \in \mathbb{N}$ such that $\text{GCD}(p, m) = 1$. How many Sylow 3-subgroups are there in a noncyclic group G of order 21? [3+2]
   (b) Find all Sylow 3—subgroups in S_4. [3]
   (c) Let p be a prime number and $n > 1$ be any integer. Does there exists a simple group of order p^n? Justify your answer. [2]

3. (a) Show that the symmetric group S_n on n symbols is not solvable for $n \geq 5$. [5]
   (b) Write a solvable series for the alternating group A_4. [2]
   (c) Let G be a finite group and H be a subgroup of G. Let $S = \{aH : a \in G\}$. Show that there exists a homomorphism ϕ from G into the set of all permutations on S such that $\text{Ker}\phi \subseteq H$. [3]

4. (a) Prove that $\mathbb{Z}_4[x]$ has has infinitely many units. [3]
   (b) Let E be an Euclidean domain. Let $a, b, q, r \in E$ such that $b \neq 0,\ a = bq + r$. Show that $GCD(a, b) = GCD(b, r)$. [3]
   (c) Is $\mathbb{Z}[x, y]/ < 2x >$ an integral domain? Justify your answer. [2]
   (d) Is $2|2$ possible in the ring $2\mathbb{Z}$? Justify your answer. [2]

5. (a) Show that every principal ideal domain is a unique factorization domain. Prove or disprove that the integral domain $\mathbb{Z}[i\sqrt{5}]$ is not a unique factorization domain. [5+1]
   (b) Show that the ring $\mathbb{Z}[i]$ of Gaussian integers is a factorization domain. Check whether $2 + 3i$ is a prime element in $\mathbb{Z}[i]$ or not? Justify your answer. [2+2]

6. (a) Show that $< x^6 + x^3 + 1 >$ is a maximal ideal in $\mathbb{Q}[x]$. [3]
   (b) Find all irreducible polynomials of degree 2 in $\mathbb{Z}_2[x]$. [3]
   (c) Let R be a principal ideal domain. Show that a non zero proper ideal P is prime if and only if P is generated by a prime element of R. [2+2]

M. Sc. Examination-2024

Semester-II

Mathematics

Course: MMC-24 (Classical Mechanics)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. The coordinate system $Oxyz$ is rotating with angular velocity $\vec{\omega}(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \sin^2 t\,\hat{k}$ relative to a fixed (inertial) frame of reference $OXYZ$. Find $\frac{d\vec{A}}{dt}(t)$ and $\frac{d^2\vec{A}}{dt^2}(t)$ relative to the moving $(Oxyz)$ and fixed frame of references $(OXYZ)$ for $\vec{A}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath} + e^t\,\hat{k}$. Also find the Coriolis and centripetal part of $\frac{d^2\vec{A}}{dt^2}(t)$ at $t = 2$. [10]

2. A symmetric rigid body is rotating under the action of an external torque $\vec{\Lambda}$. Derive the evolution equation for the angular velocity $\vec{\omega}(t)$ for its rotational motion. Verify whether $|\vec{\omega}(t)|$ changes with time or not if $\vec{\Lambda} = 0$. Examine whether $\vec{\omega}(t), \vec{\Omega}(t)$ and the axis of symmetry of the symmetric rigid body lie on a plane throughout the motion in case of free motion of the rigid body. [10]

3. State Hamilton's principle and derive the condition for a set of new functions of phase space variables would be a new set of canonical variables. Find the values of α, β for which the new variables $(Q(q,p), P(q,p)) = (q^\alpha \cos(\beta\, p), q^\alpha \sin(\beta\, p))$ are canonical. [10]

4. Discuss the stable, unstable and the dynamic equilibrium of mechanical system. Explain normal coordinates and normal frequency of oscillations of a mechanical system around close vicinity of one of its stable equilibrium state. Find the equilibrium point, normal coordinates and normal frequency of a mechanical system with the Lagrangian

$$L = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) - \left\{\frac{k_1}{2}(q_1 - a)^2 + \frac{k_2}{2}(q_2 - b)^2 + \frac{k_3}{2}(q_3 - c)^2\right\},$$

if exist. [10]

5. Define Poisson bracket of physical observables which are functions of phase space variables. Write down the evolution equation for any physical observables in terms of Poisson bracket. Define the constant of the motion. Prove that Poisson bracket of two constants of the motion is also a constant of the motion. [10]

6. State the basic principles of Quantum mechanics. Prove that the time and the energy of a conservative system may be regarded as the pair of canonically conjugate variables. State canonical quantization rules and derive the Schrödinger equation for the description of the microscopic motion of a particle moving under the influence of force described by the potential energy function $V(\vec{r})$. Interpret the unknown function appearing in the equation. [10]

Use separate answer script for each unit

M. Sc. Examination-2024

Semester-II
Mathematics
Paper : MMC-25 (New Syllabus)
(Solid Mechanics and Dynamical Systems)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I [Solid Mechanics (Marks: 20)]

Answer *any two* questions.

1. (a) Prove that, in case of infinitesimal deformation, $E_{ij}, (i \neq j)$ denote decrease in right angle between two orthogonal line elements. [5]

   (b) The displacement field for a continuum body is given by

$$u_1 = \epsilon(X_1^2 + X_2^2 + 2),$$
$$u_2 = \epsilon(3X_1^2 + 4X_2^2),$$
$$u_3 = \epsilon(2X_1^3 + 4X_3),$$

   where $\epsilon = 0.0001$. What is the deformed position of a point originally at (1, 2, 3)? What are strain components there? [5]

2. (a) Prove that, the first strain invariant $I_1 (= E_{11} + E_{22} + E_{33})$ geometrically represents the change in volume per unit original volume. [5]

   (b) The state of stress at a point is given by

   $\tau_{ij} =$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

   (i) Find the stress vector at the point on the plane whose normal is in the direction $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$.
   (ii) Determine the magnitude of the normal and shearing stress on this plane. [2+3]

3. (a) State the principle of balance of linear momentum. Hence, obtain Cauchy's first equation of motion. [1+4]

   (b) Prove that the maximum shearing stress at any point of a continuum is equal to one-half the difference between algebraically the largest and smallest principal stresses. [5]

Unit-II (Dynamical Systems (Marks: 20))

Answer *any two* questions.

1. (a) Find the equilibrium points and plot the phase diagrams of the following equations
   (i) $\frac{dx}{dt} + \omega^2 x = 0$, (ii) $\frac{dx}{dt} - \omega^2 x = 0$
   by converting these equations into a system of ODEs. [5]

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function and p be a fixed point of f. If $|f'(p)| < 1$, then show that p is the stable fixed point. [3]

   (c) Find the period of the point $\frac{5+\sqrt{5}}{8}$ for the map $f(x) = 4x(1-x)$, $x \in [0, 1]$. Also determine its stability. [2]

2. (a) Define Lyapunov stability and orbital stability. [1+1]

   (b) Using the Lyapunov function $L(x, y, z) = x^2 + 2y^2 + z^2$, show that the origin is an asymptotically stable equilibrium point of the system $\dot{x} = -2y + yz - x^3$, $\dot{y} = x - xz - y^3$, $\dot{z} = xy - z^3$ [3]

   (c) Prove that the periodic-2 cycle of the logistic map $f(x) = \lambda x(1-x)$, $x \in [0, 1]$ is linearly stable when $3 < \lambda < 1 + \sqrt{6}$ [4]

   (d) Find the fixed point of the Henon map $H_{a,b}(x, y) = (a - x^2 + by, x)$. [1]

3. (a) What is the Lienerd system. State Lienerd theorem. Hence show that the Van-der-Pol equation has a unique stable limit cycle. [1+1+3]

   (b) Suppose $f(x)$ is a polynomial such that all the roots of $f'(x)$ are real and distinct. Then prove that the Schwarzian derivative $Sf(x) < 0$. Using the Schwarzian derivative determine the stability of the fixed point of the map $f(x) = -\sin x$ [2+1]

   (c) Find the fixed point of the Euler shift map. Show that the cycles $\{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$ and $\{\frac{3}{7}, \frac{6}{7}, \frac{5}{7}\}$ form two unstable periodic-3 cycles of the Euler shift map. [1+1]

M.Sc. Examination, 2024

Semester-II

Mathematics

Paper: MMC-26 (New & Old Syllabus)

(Numerical Analysis)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Define n-th order divided difference of $f(x)$. Show that an explicit expression for the n-th order divided difference is given by [5]

$$f[x_0, x_1, \cdots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod\limits_{\substack{j=0 \\ i \neq j}}^{n} (x_i - x_j)}.$$

(b) If S is a natural cubic spline that interpolates a twice continuously differentiable function f at the knots $a = x_0 < x_1 < \cdots < x_n = b$, then show that [5]

$$\int_a^b [S''(x)]^2 dx \leq \int_a^b [f''(x)]^2 dx.$$

2. (a) Derive Newton's backward difference formula from Generalized Newton's divided difference interpolation formula. [4]

(b) The shifted Chebyshev polynomials $\hat{T}_n(x)$ defined on $[0, 1]$ are obtained from the Chebyshev polynomials $T_n(x)$ valid on $[-1, 1]$ by the variable transformation $x = 2t-1$.

(i) Using the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, $T_0(x) = 1$, $T_1(x) = x$, show that the shifted polynomials $\hat{T}_n(x)$ satisfy the recurrence relation $\hat{T}_{n+1}(x) = (4x - 2)\hat{T}_n(x) - \hat{T}_{n-1}(x)$, $n \geq 1$.

(ii) Using the first four shifted Chebyshev polynomials, obtain the least square approximation of second degree for $f(x) = \sqrt{x}$ on $[0, 1]$.

[1+5]

3. (a) What do you mean by closed-type numerical quadrature formula? If $f \in C^{n+1}[a, b]$ then show that for $a < \eta < b$, the error committed in the closed-type Newton-Cotes quadrature formula, for n odd is

$$E_{NC} \cong \frac{h^{n+2} f^{(n+1)}(\eta)}{(n+1)!} \int_0^n u(u-1)\cdots(u-n)du.$$

[1+6]

(b) Evaluate the integral $I = \displaystyle\int_2^3 \frac{\cos 2x}{1 + \sin x} dx$ by using the Gauss-Legendre three-point formula. [3]

4. (a) Using Jacobi method, find all the eigenvalues and the corresponding eigenvectors of the following matrix: [5]

$$M = \begin{bmatrix} 1 & \sqrt{3} & 4 \\ \sqrt{3} & 5 & \sqrt{3} \\ 4 & \sqrt{3} & 1 \end{bmatrix}$$

(b) Find an LU factorization of the following matrix without using elementary matrices while finding L. [5]

$$A = \begin{pmatrix} 2 & 1 & -4 & 1 \\ -4 & 3 & 5 & -2 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & -3 & 2 \end{pmatrix}$$

5. (a) Using the Gram-Schmidt orthogonalization process, compute the first four orthogonal polynomials $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x)$ which are orthogonal on $[0,1]$ with respect to the weight function $w(x) = 1$. [3]

(b) A natural cubic spline S is defined by

$$S(x) = \begin{cases} 1 + B(x-1) - D(x-1)^3, & x \in [1,2] \\ 1 + b(x-2) - \dfrac{3}{4}(x-2)^2 + d(x-2)^3, & x \in [2,3] \end{cases}$$

If S interpolates the data $(1,1), (2,1)$ and $(3,0)$, find B, D, b and d. [4]

(c) Write proper sequence of steps in the Givens method for symmetric matrices. State the main advantage of Householder's method over Givens method. [2+1]

6. (a) Derive the following finite difference formula for the derivative of the function $u(x,y)$:

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_{j,k} = \frac{1}{2(\Delta x)^3}[u_{j+2,k} - 2u_{j+1,k} + 2u_{j-1,k} - u_{j-2,k}] + O[(\Delta x)^2].$$

[3]

(b) Derive the FTCS scheme for approximating the heat-conduction problem $u_t = \alpha u_{xx}$, $u(x,0) = f(x), 0 < x < L$ and $u(0,t) = u(L,t) = 0, \forall\, t > 0$ at the $(i,n)^{\text{th}}$ point with α being positive constant. [3]

(c) What do you mean by consistency of a finite difference scheme? Show that FTCS scheme for approximating the parabolic equation $u_t = \alpha u_{xx}$ at the $(i,n)^{\text{th}}$ point is consistent. [1+3]