Use separate answer scripts for each unit

Undergraduate Examination-2024

Semester-I
Mathematics
Course: MJMA 01
(Algebra-I and Analysis I)

Time: Three Hours **Full Marks: 80**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I: Algebra I (Marks: 40)
Answer ***any four*** questions.

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1. (a) Find the inverse of an element $\begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix}$ in $GL(2, \mathbb{Z}_{11})$. [3]
   (b) For any integer $n \geq 2$, show that there are at least two elements in U_n. thats satisficn [3]
   (c) Show that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for every $a, b \in G$. [2+2]

2. (a) Find a group that contains elements a and b such that $O(a) = O(b) = 2$ and $O(ab) = 2$. [2]
   (b) Let G be a group and $a \in G$is an element of order n. If k divides n, show that $O(a^{\frac{n}{k}}) = k$. [4]
   (c) Suppose G is a group that has exactly eight elements of order 3. How many subgroups of order 3 are there in G? [4]

3. (a) Let a be the only element of order 2 in a group G. Then show that $a \in \mathbb{Z}(G)$. [3]
   (b) Show that the set $H = \{A \in GL(2, \mathbb{R}) |\ \det A$ is a power of $2\}$ is a subgroup of $GL(2, \mathbb{R})$. [3]
   (c) List all subgroups of D_4. [4]

4. (a) Give an example of a non cyclic group, all of whose proper subgroups are cyclic. Justify your answer. [3]
   (b) Prove that a group G of order n is cyclic if and only of there is an element of order n in G. [2+2]
   (c) Suppose that a cyclic group G has exactly three subgroups: G itself, $\{e\}$, and a subgroup of order 7. Find the order of G. [3]

5. (a) Show that the number of all even permutations on a finite set is equal to the number of all odd permutations. [4]
   (b) Find $\beta \in S_7$ be such that $\beta^4 = (2143567)$. [3]
   (c) How many odd permutations of order 4 are there in S_6? [3]

6. (a) Show that every element in A_n is either a 3−cycle or a product of 3-cycles. [4]
   (b) Find a conjugate element of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 5 & 4 & 7 & 8 & 6 \end{pmatrix}$ in S_8 [3]
   (c) How many proper subgroups are there in a cyclic group of order 100? Justify your answer. [3]

Unit-II: Analysis I (Full Marks : 40)

Answer any four questions

7. (a) Prove the following results : [5]

   (i) If z and a are elements in $\mathbb{R}$ with $z + a = a$ then $z = 0$.

   (ii) If u and $b(\neq 0)$ are elements in $\mathbb{R}$ with $u \cdot b = b$ then $u = 1$.

   (iii) If $a \in \mathbb{R}$ then $a \cdot 0 = 0$.

   (b) Prove the following results: [5]

   (i) If $a \in \mathbb{R}$ and $a \neq 0$ then $a^2 > 0$.

   (ii) If $a \in \mathbb{R}$ and $a > 0$ then $-a < 0$.

8. (a) For $x \in \mathbb{R}$ and $x > 0$, prove the following results :

   (i) There exists a natural number n such that $0 < \frac{1}{n} < x$.

   (ii) There exists a natural number m such that $m - 1 \leq x < m$. [4]

   (b) Show that countable union of countable sets is countable. [6]

9. (a) Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be three sequences such that $x_n \leq y_n \leq z_n \; \forall n \in \mathbb{N}$. If $x_n \to l$ and $z_n \to l$ then prove that $\{y_n\}$ is convergent and converges to l. [4]

   (b) Show that $\lim\limits_{n\to\infty} \{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + ... + \frac{1}{(n+n)^2}\} = 0$. [4]

   (c) Evaluate $\lim\limits_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$. [2]

10. (a) Define limit point of a set. Show that 0 is a limit point of the set $\{1, \frac{1}{2}, \frac{1}{3}, ...\}$ [3]

    (b) Let $S \subset \mathbb{R}$ and x be a limit point of S. Prove that every neighborhood of x contains infinitely many points of S. [3]

    (c) Give the $(\epsilon - \delta)$ definition of limit of a function and hence evaluate $\lim\limits_{x\to 2} f(x)$ where $f(x) = x^3 - 4x^2 + 3x + 2, \; x \in [1, 4]$. [4]

11. (a) When is a sequence said to be convergent ? Prove that every convergent sequence is bounded. Does the converse hold ? Justify your answer. [5]

    (b) Evaluate : $\lim\limits_{n\to\infty} (1 + \frac{1}{n})^n, \; n \in \mathbb{N}$. [5]

12. (a) State and prove Cauchy's First Limit Theorem. [5]

    (b) Evaluate : $\lim\limits_{n\to\infty} \frac{1}{n}\{(2n+1)(2n+2)...(2n+n)\}^{\frac{1}{n}}$ [5]

Use separate answer script for each unit

Four Year Undergraduate Examination-2024

Semester-I
Mathematics
Course: MJMA 02
(Analytical Geometry and Vector Calculus)

Time: Three Hours **Full Marks: 80**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit I: Analytical Geometry (Full Marks: 40)

Answer *any four* questions

1. (a) If transformation from one set of rectengular axes to another with same origin the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ changes to $a'X^2 + 2h'XY + b'Y^2 + 2g'X + 2f'Y + c'$ then prove that
   (*i*) $a'b' - h'^2 = ab - h^2$ (*ii*) $a' + b' = a + b$. [6]

   (b) Find the polar equation of the straight line joining two points on the parabola $\frac{2a}{r} = 1 + Cos\theta$ with $\alpha - \beta$ and $\alpha + \beta$ as their vectorial angles. [4]

2. (a) Reduce the equation $3x^2 + 2xy + 3y^2 - 16x + 20 = 0$ to its canonical form. [6]

   (b) Find the condition that the straight line $rCos(\theta - \alpha) = p$ touches the conic $\frac{l}{r} = 1 + eCos\theta$ [4]

3. (a) Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0,\ 2x + 3y + 4z = 8$ is a great circle. [4]

   (b) A sphere of constant radius r passes through the origin O and cuts the axes in A,B,C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is given by $(x^2 + y^2 + z^2)^2(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}) = 4r^2$. [6]

4. (a) Find the equation of the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + cz^2 = 1$. [4]

   (b) Find the condition that the plane $ax + by + cz = 0$ $(abc \neq 0)$, cuts the cone $yz + zx + xy = 0$ in perpendicular straight lines. [6]

5. (a) A variable plane through x-axis and a variable plane through y-axis are inclined at a constant angle α. Prove that their line of intersection generates the cone $z^2(x^2 + y^2 + z^2) = x^2y^2(tan\alpha)^2$. [5]

   (b) Normals are drawn from the point (α, β, γ) to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; prove that if the feet of the three normals lie on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, then the feet of the remaining three will lie on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + 1 = 0$. [5]

6. (a) Prove that any two generators of the different systems of hyperboloid of one sheet intersect. [5]

   (b) Find the equations of the generators of the hyperboloid $x^2 + y^2 - 4z^2 = 9$ passing through the point $(-3, 4, 2)$. [5]

Unit-II : Vector Calculus (Full Marks: 40)

Answer *any four* questions

1. (a) Using vector method, prove that the straight lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent.

(b) Prove, using vector method, that the internal bisectors of the angles of a triangle are concurrent.

(c) Let $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ be three vectors such that $\vec{\alpha}$ and $\vec{\gamma}$ are perpendicular to each other. Justify, with proof, whether the vectors $\vec{\alpha} \times (\vec{\beta} \times \vec{\gamma})$ and $(\vec{\alpha} \times \vec{\beta}) \times \vec{\gamma}$ are perpendicular to each other. [4+4+2]

2. (a) Let $\vec{a}$ and $\vec{b}$ be two given vectors. Find vectors $\vec{x}$ and $\vec{y}$ such that $\vec{x} + \vec{y} = \vec{a}$, $\vec{x} \times \vec{y} = \vec{b}$ and $\vec{x} \cdot \vec{a} = 1$.

(b) Find the value of p such that the vectors $\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$, $2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and $3\hat{\imath} + 2\hat{\jmath} + p\hat{k}$ are coplanar.

(c) Prove that the straight lines $\vec{r}_1 = \vec{a} + t(\vec{b} + \vec{c})$ and $\vec{r}_2 = \vec{b} + s(\vec{c} + \vec{a})$ intersect.

(d) Prove that the straight lines $\vec{r}_1 = -3\vec{a} + 6\vec{b} + t(-4\vec{a} + 3\vec{b} + \vec{c})$ and $\vec{r}_2 = -2\vec{a} + 7\vec{c} + s(-4\vec{a} + \vec{b} + \vec{c})$ do not intersect. [3+3+2+2]

3. (a) State and prove the Frenet-Serret formulae.

(b) For the differentiable curve $\vec{R}(t) = e^t \cos t\ \hat{\imath} + e^t \sin t\ \hat{\jmath} + e^t\ \hat{k}$, find the torsion τ.

(c) Using Frenet-Serret formulae, prove that $-\frac{d\vec{T}}{ds} \cdot \frac{d\vec{B}}{ds} = \kappa\tau$. [5+3+2]

4. (a) Determine the value of λ so that the vector field $\vec{F}(x, y, z) = (x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x + \lambda z)\hat{k}$ is solenoidal.

(b) Prove that the vector field $y\ \hat{\imath} + x\ \hat{\jmath}$ is both irrotational and solenoidal.

(c) Determine whether the vector field $\sin y\ \hat{\imath} + \sin x\ \hat{\jmath} + e^z\ \hat{k}$ is solenoidal or irrotational. [3+3+4]

5. (a) Define orientation-preserving and orientation-reversing parametrizations of a differentiable curve. Give an example in each case.

(b) Prove that the scalar line integral of a scalar function over a parametrized curve does not depend on the orientation of the curve.

(c) Prove that every vector line integral can also be expressed as a scalar line integral of some scalar function. [4+4+2]

6. (a) Verify using Green's theorem that the area of the rectangle $[0, a] \times [0, b]$ is ab.

(b) Using Green's theorem, determine the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(c) Let $\vec{r}(t) : [0, 1] \to \mathbb{R}^3$ be a differentiable curve such that $\vec{r}(t)$ is an injective function on $(0, 1)$ and $\vec{r}(0) = \vec{r}(1)$. Prove that $\int_{\vec{r}} 3x^2y\,dx + x^3dy = 0$. [3+3+4]

Undergraduate Examination-2024

Semester-I

Mathematics

Course : SECMA 01 (Algebra II and Analysis II)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

UNIT - I : Algebra II (Marks : 40)

Answer *any four* questions.

1. (a) Define reduced row echelon matrix. Reduce the augmented matrix of the system

$$2x + y - 3z = 5$$
$$x - 7z = 8$$
$$5x - 2y + 3z = 2$$

to its reduced row echelon form. [2+2]

(b) Solve the following system.

$$x - 2y + 3z = 1$$
$$2x - 5y + 10z = 0$$
$$4x - 7y + 8z = 2.$$

[3]

(c) Find the values of k so that the following system has unique solution.

$$x + y + z = 1$$
$$x + 2y + 4z = k$$
$$x + 4y + 10z = k^2.$$

[3]

2. (a) Draw the column picture for the system $2x + y = 8$ and $x + 2y = 7$. [3]

(b) Determine if $(2, -1, 6)$ is a linear combination of the vectors $(1, -1, 2), (5, -4, 4)$ and $(3, -2, 8)$. [3]

(c) Express the vector $(2, 5, 7)$ as a linear combination of the three vectors $(1, 2, 3), (1, 0, 1)$ and $(1, 3, 4)$. [4]

3. (a) Find the value of h such that the vector $(1, h, 1)$ is in the plane spanned by the vectors $(3, 5, 4)$ and $(2, 4, 3)$. [3]

(b) Show that two vectors in $\mathbb{R}^3$ can not span the whole $\mathbb{R}^3$. [3]

(c) Define linear independence of vectors. Check linear independence of the vectors $\{(1, 0, -1, 2), (-3, 2, -5, 4), (5, 7, -4, 3)\}$. [1+3]

4. (a) Find the values of h so that $\{(1, -2, -1, 3), (-3, 0, 5, 4), (-5, 6, -4, h)\}$ is linearly independent. [3]

(b) Let $\alpha, \beta \in \mathbb{R}^3$. Show that α and β are linearly independent if and only if they span a plane through the origin. [4]

(c) Define subspaces of $\mathbb{R}^n$. Show that $V = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 3y - z = 0\}$ is a subspace of $\mathbb{R}^3$. [3]

5. (a) Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}^4$ and $A = (\alpha_1|\alpha_2|\alpha_3|\alpha_4)$. Show that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis of $\mathbb{R}^4$ if and only if A is invertible. [4]

(b) Find the null space of the matrix $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 1 \\ 3 & 2 & 11 \end{pmatrix}$ and a spanning set of the null space. [3]

(c) Let $A \in M_3(\mathbb{R})$ and $b \in \mathbb{R}^3$. If the system $AX = b$ has a unique solution, then show that $r(A) = 3$. Hence or otherwise show that the columns of A span $\mathbb{R}^3$. [3]

6. (a) Show that every subspace of $\mathbb{R}^n$ has a basis. [4]

   (b) Find a basis and the dimension of the subspace $V = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 3y - z = 0\}$. [3]

   (c) Find a basis of the solution space of the homogeneous system:

$$\begin{aligned} x - 2y + z &= 0 \\ 2x + 3y - 4z &= 0 \\ 4x - y - 2z &= 0. \end{aligned}$$

[3]

Unit-II : Analysis II (Marks : 20)

Answer *any two* questions.

1. (a) Find y_n where $y = e^x sin^2 x$. [3]

   (b) If $y = Sin^{-1}x$ then show that

   (i) $(1 - x^2)y_2 - xy_1 = 0$.

   (ii) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2 y_n = 0$.

   Find also the value of $(y_n)_0$. [7]

2. (a) Evaluate $\lim_{x \to 0} (\frac{1}{x^2})^{tan x}$. [4]

   (b) Find the values of a and b in order that

$$\lim_{x \to 0} \frac{x(1 + aCosx) - bSinx}{x^3} = 1.$$

[6]

3. (a) State and prove Rolle's theorem. [5]

   (b) Verify Rolle's theorem for the function f defined in $[1,2]$ by $f(x) = x(x - 1)(x - 2)$. [2]

   (c) In the Mean Value Theorem, $f(x + h) = f(x) + hf'(x + \theta h)$, if $f(x) = a + bx + cm^x$ where a, b, c and m are constants, then show that θ is independent on x. [3]