Use separate answer script for each group

Undergraduate Examination-2024

Semester-II
Mathematics
MJMA03 (Algebra III)

Full Marks: 80

Time: Four Hours

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any *eight* questions, by choosing *four* questions from each group

Group-A(Group Theory)

Answer *any four* questions. $(4 \times 10 = 40)$

1. (a) For $n \geq 3$, let $H = \{\beta \in S_n | \beta(1) = 1 \text{ or } 2 \text{ and } \beta(2) = 1 \text{ or } 2\}$. Prove that H is a subgroup of S_n. Determine $|H|$ for $n = 5$. [3+1]
   (b) Explain why S_8 contains subgroups isomorphic to $\mathbb{Z}_{15}$ and $U(16)$? [2+2]
   (c) Let $\beta \in S_7$ and suppose $\beta^4 = (2143567)$. Find β. [2]

2. (a) State and Prove Lagrange's Theorem on finite groups. [1+3]
   (b) Let H be a subgroup of $\mathbb{R}^*$, the group of nonzero real numbers under multiplication. If $\mathbb{R}^+ \subseteq H \subseteq \mathbb{R}^*$, Then prove that $H = \mathbb{R}^+$ or $H = \mathbb{R}^*$. [3]
   (c) Let $|G| = 33$. What are the possible orders for the elements of G? Show that G must have an element of order 3. [2+1]

3. (a) Let G be a group of order 100 that has a subgroup H of order 25. Show that every element of G of order 5 is in H. [4]
   (b) Prove that an Abelian group of order 15 is cyclic. [3]
   (c) Let H and K be subgroups of a finite group G with $H \subseteq K \subseteq G$ and $[G : H]$ a prime number. Show that either $H = K$ or $K = G$. [3]

4. (a) Does the subgroup $H = \{(1), (12)(34)\}$ normal in A_4? Justify your answer. [2]
   (b) What is the order of the element $14+ < 8 >$ in the factor group $\mathbb{Z}_{24}/ < 8 >$? Give an example of non commutative group that have a cyclic quotient group. Justify your answer. [2+1]
   (c) If a factor group $G/Z(G)$ has order p^2 for some prime p, to what familiar group is $G/Z(G)$ isomorphic? Explain why there is no group G with the property that $G/Z(G)$ is isomorphic to the group of integers under addition. [3+2]

5. (a) Show that subgroup of a group with index 2 is normal. Give an example of a normal subgroup in group with index greater than 2.Justify your answer. [2+3]
   (b) Show that every subgroup of an Abelian group is normal. Give an example with proper justification of a non commutative group every subgroup of which is normal. [2+3]

6. (a) State and prove First Isomorphism Theorem on groups. [1+4]
   (b) Suppose that ϕ is a homomorphism from S_4 onto $\mathbb{Z}_2$. Determine Ker ϕ. [2]
   (c) Find all homomorphisms from $\mathbb{Z}_6$ to S_3. [3]

Group-B(Ring Theory)

Answer ***any four*** questions. $(4 \times 10 = 40)$

7. (a) Describe the units of $M_2(\mathbb{Z})$. [3]
   (b) Let R be a ring. If $(R, +)$ is a cyclic group, show that R is a commutative ring. [3]
   (c) Let R be a commutative ring with unity. If a_o is a unit and $a_1, a_2, \cdots, a_n$ are nilpotents in R, show that $a_o + a_1x + a_2x^2 + \cdots + a_nx^n$ is a unit in $R[x]$. [4]

8. (a) Give an example of an infinite Boolean ring with justification. [3]
   (b) Prove that $\mathbb{Z}_n$ is a field if and only if p is a prime number. [4]

(c) Show that every Boolean ring is commutative. [3]

9. (a) Let R be a finite commutative ring with unity. Show that every nonzero element of R is a unit or a zero-divisor. [4]

(b) Show that the set of all nilpotent elements of a commutative ring form an ideal. [3]

(c) List all the nilpotent elements of $\mathbb{Z}_{100}$. Find the total number of nilpotent elements of $\mathbb{Z}_{p^m q^n}$ where p, q are prime numbers and m, n are positive integers. [3]

10. (a) Give an example with justification of an infinite integral domain that has characteristic 3. [3]

(b) Does there exist any subring of $\mathbb{Z} \times \mathbb{Z}$ that is not an ideal of $\mathbb{Z} \times \mathbb{Z}$? Show that every subring of $\mathbb{Z}$ is an ideal of $\mathbb{Z}$. [4]

(c) Let R be a ring with 1 and has no divisors of zero. Show that for every $a, b \in R$, $ab = 1$ if and only if $ba = 1$. [3]

11. (a) Show that the factor ring $\mathbb{R}[x]/ < x^2 + 1 >$ is a field. [3]

(b) Determine all ideals of $\mathbb{R} \times \mathbb{R}$. [3]

(c) Let m and n be two positive integers and $d = \gcd(m, n)$. Show that $< m, n > = < d >$. [4]

12. (a) Prove that the ideal $I = \{(3x, y) \mid x, y \in \mathbb{Z}\}$ is not properly contained in any proper ideal of $\mathbb{Z} \times \mathbb{Z}$. [4]

(b) Give an example of a ring that has exactly two proper nontrivial ideals. Justify. [3]

(c) Prove that the ring $M_n(\mathbb{R})$ of all $n \times n$ real matrices has no nontrivial proper ideals. [3]

Undergraduate Examination-2024

Semester-II

Mathematics

Core Course: MJMA04

(Differential Equations-I)

Full Marks: 80

Time: Three Hours

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer Question No. 1 and any *ten* from the rest.

[$10 \times 2 = 20$]

1. Answer any *ten* questions from the following.
   (a) Find the non-zero value of n for which the differential equation $(3xy^2 + n^2x^2y)dx + (nx^3 + 3x^2y)dy = 0$ is an exact differential equation. [2]
   (b) Solve the differential equation $3x(xy-2)dx + (x^3+2y)dy = 0$. [2]
   (c) Find an integrating factor of the differential equation $\frac{dy}{dx} = \frac{2xy^2+y}{x-2y^3}$. [2]
   (d) Find the orthogonal trajectories of the family of parabolas $y = cx^2$. [2]
   (e) Solve the nonlinear differential equation $(\frac{dy}{dx})^2 + (e^x + e^{-x})\frac{dy}{dx} + 1 = 0$. [2]
   (f) Solve: $(D^2+9)^3(D^2-16)^3y = 0$. [2]
   (g) Solve the differential equation $\frac{d^3y}{dx^3} = xe^x$. [2]
   (h) Find a second order linear homogeneous differential equation whose linearly independent solutions are x, and xe^x. [2]
   (i) Solve the boundary value problem $\frac{d^2y}{dx^2} + y = 0, y(0) = 0, y(\frac{\pi}{2}) = 1$. [2]
   (j) Find two different solutions of the differential equation $\frac{dy}{dx} = y^{1/3},\ y(0) = 0$. [2]
   (k) Solve: $\frac{d^2y}{dx^2} = 8y^3$. [2]
   (l) Find the singular solution of the differential equation $y = px + p^2$. [2]
   (m) Solve: $\frac{d^4y}{dx^4} - cot x\frac{d^3y}{dx^3} = 0$. [2]
   (n) Solve the differential equation $\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 + 1 = 0$. [2]
   (o) Solve: $\frac{d}{dx}(x\frac{dy}{dx}) = x;\ y(1) = 0, \frac{dy}{dx}|_{x=1} = 0$. [2]

2. Obtain the differential equation of family of circles $x^2+y^2+2gx+2fy+c = 0$, where g, f and c are parameters. [6]

3. Solve the homogeneous equation $(x^2 - xy + y^2)dx - xydy = 0$. [6]

4. If $e^{\int \phi(x)dx}$ is an integrating factor of the differential equation $M(x,y)dx + N(x,y)dy = 0$, then find the expression of $\phi(x)$. Hence solve the equation $2ydx + xdy = 0$. [4+2]

5. Show that if f is any solution of the differential equation $\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$, then the transformation $y = f + \frac{1}{v}$ reduces the above differental equation to a linear equation in v. Hence solve the equation $\frac{dy}{dx} = (1-x)y^2 + (2x-1)y - x$; given solution $f(x) = 1$. [4+2]

6. Prove that the transformation $v = y^{1-n}(n \neq 0\ or\ 1)$ reduces the Bernoulli equation $\frac{dy}{dx} = P(x)y + Q(x)y^n$ to a linear equation in v. Hence solve the equation $\frac{dy}{dx} + y = xy^3$. [3+3]

7. Solve the differential equation $\frac{dy}{dx} + y = f(x)$ with initial condition $y(0) = 0$, where $f(x) = 2$, when $0 \leq x < 1$ and $f(x) = 0$, when $x \geq 1$. Hence find the value of $y(5)$. [5+1]

8. Find the orthogonal trajectories of the family of curves $r^n sin(n\theta) = c^n$, c being the parameter. [6]

9. Find a family of oblique trajectories that intersect the family of straight lines $y = cx$ at angle 45^o. [6]

10. Solve the nonlinear differential equation $y^2 log(y) = xyp + p^2$. [6]

11. Reduce the equation $(px - y)(x - py) = 2p$ to Clairaut's form by the substitution $x^2 = u$ and $y^2 = v$. Hence solve the equaion. [5+1]

12. Solve the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$. [6]

13. Solve: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^{2x} sinx$. [6]

14. Solve the Cauchy-Euler equation $x^2y'' - 4xy' + 6y = log(x^2)$, $x > 0$. [6]

15. Using the method of variation of parameters, solve the equation $(D^2 + 1)y = secx\ tanx$. [6]

16. Test for exactness and solve the differential equation $(1 + x^2)y'' + 4xy' + 2y = sec^2x$; given that $y(0) = 0$ and $y'(0) = 1$. [2+4]

Use separate answer script for each unit

Undergraduate Examination-2024

Semester-II (NEP)
Mathematics
Course: SECMA - 02
(Analysis-II and Tensor Calculus)

Time: 3 Hours Full Marks: 60

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit I: Analysis - II (Full Marks: 20)

Answer *any two* questions

1. (a) What do you mean by the limit of a function $f(z)$ when z approaches z_0? Explain with examples. Show that such a limit, if exists, is unique.

(b) Prove, using ϵ-δ notations, that $f(z) = \frac{1}{\bar{z}}$ is continuous at $z = i$.

(c) Find the stereographic projection of the straight line $19x+8y = 24$ on the Riemann sphere $\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = \frac{1}{4}$. [(2+3)+3+2]

2. (a) Use ϵ-δ notations to prove $\lim_{z\to -1} \frac{z^2+7z+9}{2z+5} = 1$.

(b) When does a complex sequence $\{z_n\}$ converge to a limit z? Explain with examples. Find the limit points of the sequence $\{i^n + \frac{1}{n}\}$.

(c) Describe, in detail, how a point $z = x+iy$ on the complex plane is stereographically projected onto a point (ξ, η, ζ) on a suitable sphere. [3+(2+2)+3]

3. (a) Examine whether the limit $\lim_{z\to 0} \frac{\bar{z}^2}{z}$ exists. Hence find the value of $\lim_{z\to 0} \frac{\bar{z}^2 e^{\frac{i}{|z|}}}{z}$.

(b) If $f(z)$ and $g(z)$ are continuous at z_0 and $c \in \mathbb{C}$, show that $cf(z)+g(z)$ is continuous at z_0.

(c) Prove that a convergent sequence is bounded. [(2+2)+3+3]

Unit-II : Tensor Calculus (Full Marks: 40)

Answer *any four* questions

1. (a) Define dummy index and free index in an indexed expression. Using summation convention, expand $\delta^i_j a^{jk}$ and prove that it equals a^{ik}.

(b) Define e-systems of order 2 and prove that $e_{ij}e^{ik} = \delta^j_k$.

(c) If $a_{pq}x^p x^q = 0$ for all values of the independent variables $x^1, \ldots, x^n$ and a_{pq}'s are constants, then prove that $a_{ij} + a_{ji} = 0$. [3+3+4]

2. (a) If $x^i = a^i_p y^p$ and $z^i = b^i_q x^q$, then prove that $z^i = b^i_p a^p_q y^q$.

(b) Using summation convention, compute $\delta^i_k \delta^k_\ell \delta^\ell_i$ and $\delta^i_j \delta^j_\ell \delta^\ell_k \delta^k_i$.

(c) Prove that $e_{imn}e^{irs}a^{mn} = a^{rs} - a^{sr}$. [2+4+4]

3. (a) If A^i and B_i are respectively contravariant and covariant vectors, then prove that $A^i B_i$ is an invariant.

(b) If A^i and B^j are two contravariant vectors, then prove that the n^2 quantities $A^i B^j$ are the components of a contravariant tensor of rank 2.

(c) Define mixed tensors of rank 3 with first order of contravariance and second order of covariance and vice versa. [3+3+4]

4. (a) If the co-ordinates of a tensor are all zero in one co-ordinate system, then prove that they are all zero in every other co-ordinate system.

(b) Prove that the outer multiplication of two mixed tensors of types (p, q) and (r, s) is a mixed tensor of type $(p + r, q + s)$.

(c) Let $p \geq 6$ and $q \geq 3$ be two integers. Can you obtain a tensor of type $(p-5, q-2)$ from a tensor of type (p, q) via finite number of contractions? Justify your answer. [3+4+3]

5. (a) Let A_i be a covariant vector, determine, with proof, whether $\dfrac{\partial A_i}{\partial x^j}$ is a tensor or not.

(b) If a_{ij} is a skew-symmetric tensor, then prove that $(\delta^i_j \delta^k_\ell + \delta^i_\ell \delta^k_j)a_{ik} = 0$.

(c) If a tensor T_{ijk} is symmetric in the first two indices and skew-symmetric in the last two indices, then prove that $T_{ijk} = 0$. [4+3+3]

6. (a) Prove that the scalar product of a relative covariant vector of weight ω and a relative contravariant vector of weight ω' is a relative scalar of weight $(\omega + \omega')$.

(b) Prove that the determinant of a tensor of type $(1, 1)$ is an invariant.

(c) Determine, with proof, the number of distinct components of a completely symmetric tensor of type $(0, 3)$ where the indices assume values from 1 to n. [3+3+4]