Four Year Undergraduate Examination - 2024

Semester-III

Mathematics

Paper: M.JMA-5

(Analysis-III)

Time: 3 Hours **Full Marks: 80**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any eight* questions.

1. (a) Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{R}$. [6]

   (b) Examine the convergence of the series $a + b + a^2 + b^2 + a^3 + b^3 + \cdots$, where $0 < a < b < 1$. [4]

2. (a) Prove or disprove: There exists a convergent series $\sum a_n$ such that $\sum a_n^2$ is divergent. [4]

   (b) Describe Cauchy's condensation test for positive series. Use it to discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, where $p > 0$. [2+4]

3. (a) If $\{n^2 a_n\}$ is a convergent sequence, show that the series $\sum a_n$ is absolutely convergent. [4]

   (b) State Leibnitz's test and Dirichlet's test for infinite series. Give application of these. Show that Leibnitz's test is a particular case of Dirichlet's test. [2+2+2]

4. (a) Discuss in how many ways a function f may be discontinuous at a point $x = c$. Explain with examples. [4+2]

   (b) If f and g are two function continuous at c, show that $f \cdot g$ is also continuous at c. [4]

5. (a) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $\mathbb{R}$. Show that $A = \{x \in \mathbb{R} : f(x) = g(x)\}$ is a closed set in $\mathbb{R}$ [4]

   (b) If f is continuous on a closed and bounded interval $I = [a, b]$, show that $f(I)$ is a closed and bounded interval. Is $f(I)$ an open and bounded interval provided I is so? Justify your answer. [4+2]

6. (a) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be continuous on $[0, 1]$ and let f assume only irrational values. If $f(\frac{1}{e}) = e$, show that $f(x) = e$ for all $x \in [0, 1]$. [4]

   (b) If f is uniformly continuous on a bounded interval I, show that f is bounded on I. Is it true if f is continuous but not uniformly continuous on I? Justify your answer. [4+2]

7. (a) Let a function f be continuous on an open bounded interval (a, b) and have continuous extension to $\mathbb{R}$. Prove that f is uniformly continuous on (a, b). [4]

   (b) Define a Lipschitz function on $[a, b]$. If $f : [a, b] \longrightarrow \mathbb{R}$ is a Lipschitz function on $[a, b]$, show that f is a function of bounded variation on $[a, b]$. Is the converse true? Justify. [1+3+2]

8. (a) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a BV-function on $[a, b]$ and Q be a refinement of a partition P of $[a, b]$. Prove that $V(P, f) \leq V(Q, f)$. [4]

   (b) If $f(x) = [x]$, $x \in [1, 3]$, show that f is a function of bounded variation on $[1, 3]$. Find the variation function V on $[1, 3]$. Express f as the difference of two monotone increasing functions on $[1, 3]$. [2+2+2]

9. (a) If f is a bounded function on $[a, b]$ and P, Q are two partitions of $[a, b]$, show that $L(P, f) \leq U(Q, f)$ and $L(Q, f) \leq U(P, f)$. [4]

   (b) Define lower and upper Riemann integrals of bounded function on a closed and bounded interval. Find these for the function $f(x) = x^2$, $x \in [a, b]$, $a > 0$ and determine if f is Riemann integrable on $[a, b]$. [2+2+2]

10. (a) If f is a Riemann integrable function on $[a, b]$ and $f(x) > 0$ for all $x \in [a, b]$, show that f may not be Riemann integrable on $[a, b]$. Under what additional condition, f is necessarily Riemann integrable on $[a, b]$? Establish it. [3+4]

    (b) Let $f : [a, b] \longrightarrow \mathbb{R}$ be bounded on $[a, b]$ and be integrable on $[c, b]$ for every $c \in [a, b]$. Prove that f is integrable on $[a, b]$. [3]

11. (a) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 + x^3 & \text{if } x \text{ is rational} \\ x + x^2 & \text{if } x \text{ is irrational} \end{cases}$. Find $\underline{\int_0^1} f$ and $\overline{\int_0^1} f$. Hence determine if f is integrable on $[0, 1]$ [3+3+1]

(b) Examine the convergence of the improper integral $\int_0^1 \frac{x^{p-1}}{1+x}$. [3]

12. (a) When is an improper integral $\int_a^\infty f(x)dx$ called convergent? Explain with examples. [3]

(b) Show that the improper integral $\int_a^\infty \frac{dx}{x^\alpha}$ (where $a > 0$) is convergent if and only if $\alpha > 1$. Use it to show that $\int_0^\infty x^{m-1}e^{-x}dx$. is convergent if $m > 0$. [3+4] se separate answer script for each unit

Four Year Undergraduate Examination-2024

Semester-III (NEP)
Mathematics
Paper: MJMA 06
(Differential Equations II and Dynamics of a Particle)

Time: Three Hours **Full Marks: 80**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

(Unit I: Differential Equations II)
(Full Marks: 40)

Answer question number 1 and *any two* from the rest.

1. Answer *any five* questions.

(a) Solve the system of ordinary differential equations (ODEs)

$$\frac{dx}{dt} = -wy,$$
$$\frac{dy}{dt} = wx$$

and show that the point (x, y) lies on a circle.

(b) Given the solution $y_1(x) = e^x$, find the second linearly independent solution $y_2(x)$ of the ODE

$$(x+2)\frac{d^2y}{dx^2} - (4x+9)\frac{dy}{dx} + (3x+7)y = 0, \ x \in \mathbb{R} - \{2\}.$$

(c) Construct two first-order differential equations from the following ODE:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2025\, y = 0$$

with $y = 2023$ and $\frac{dy}{dx} = 2024$ at $x = 0$.

(d) If $y = u(x)$ is a solution of $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$, then show that another independent solution $y = v(x)$ is given by $v(x) = u(x)\int \frac{W(u,v)}{[u(x)]^2}dx$, where the Wronskian $W(u, v)$ satisfies the ODE $\frac{dW}{dx} + PW = 0$.

(e) Define ordinary and singular points of a second order linear ODE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = a_3(x).$$

about the point $x = x_0$. Give examples in support of your answer.

(f) Find the roots of the indicial equation of

$$2x^2\frac{d^2y}{dx^2} - (x+x^2)\frac{dy}{dx} + (x^2-2)y = 0$$

about the point $x = 0$.

(g) State the condition for integrability of the Pfaffian differential equation

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0.$$

What is the number of independent conditions of integrability in a Pfaffian differential equation with four variables?

(h) Solve the following total differential equation by inspection method:

$$(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0.$$

[5 × 2=10]

2. (a) State and prove Abel-Liouville's formula for the n^{th} order linear ODE.

(b) Let $y_1(x)$ and $y_2(x)$ be two solutions of the ODE

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\sec x)y = 0, \ -1 < x < 1$$

with Wronskian $W(x)$. If $y_1|_{x=\frac{1}{2}} = 2$, $\frac{dy_1}{dx}|_{x=\frac{1}{2}} = \frac{1}{3}$, $y_2|_{x=\frac{1}{2}} = 5$, $\frac{dy_2}{dx}|_{x=\frac{1}{2}} = 1$, find the value of $W(\frac{\sqrt{3}}{2})$.

(c) Factorize the operator on the left-hand side of $[(x+3)D^2 + (2x+7)D + 2]y = (x+3)^2e^x$ $(D^n \equiv \frac{d^n}{dx^n})$ to represent the ODE in right order.

[7+5+3]

3. (a) Solve the ODE

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$$

in series near the ordinary point $x = 0$. What will be the particular solution for $y|_{x=0} = 1$ and $\frac{dy}{dx}|_{x=0} = 2$?

(b) Solve the following system of two linear homogeneous differential equations with constant coefficients using eigenvalues and eigenvectors of the matrix:

$$\begin{aligned}
\frac{dx}{dt} &= -x - 6y, \\
\frac{dy}{dt} &= 3x + 5y, \\
x(t=0) &= 0, \ \ y(t=0) = 2.
\end{aligned}$$

[9+6]

4. (a) Identify the number of independent arbitrary constants within the general solution of the following simultaneous equations:

$$\begin{aligned}
\phi_1(D)x + \phi_2(D)y &= \phi(t), \\
\psi_1(D)x + \psi_2(D)y &= \psi(t), \ \ D \equiv \frac{d}{dt}.
\end{aligned}$$

(b) What do the solutions of the differential equation

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$$

represent geometrically? Hence solve the following:

$$\frac{dx}{x^2+y^2+yz} = \frac{dy}{x^2+y^2-xz} = \frac{dz}{z(x+y)}.$$

(c) Solve the total differential equation after satisfying the conditions of integrability

$$(yz-1)dx + (z-x)xdy + (1-xy)dz = 0.$$

[3+(1+5)+6]

Unit-II [Dynamics of a Particle (Full Marks: 40)]

Answer *any four* questions.

1. (a) Define central force and the centre of force. [3]

   (b) A particle moves from rest in a straight line under an attractive force $\mu\times$ (distance)$^{-2}$ per unit mass. Show that if the initial distance of the particle from the centre of force be $2a$, then the distance will be a after a time $\left(\frac{\pi}{2}+1\right)\left(\frac{a^3}{\mu}\right)^{1/2}$. [7]

2. (a) Prove that the resultant of two simple harmonic motions having slightly different periods is simple harmonic, of period equal to either of the component motion with varying amplitude and epoch. [3]

   (b) A mass is suspended from a ceiling by a light elastic string of natural length l. When the mass hangs in equilibrium, the length of the string is $(l+c)$. The mass is started off from the position of equilibrium with downward vertical velocity v. If, in the subsequent motion, the string never becomes slack, then show that $v^2 < cg$. [7]

3. (a) Establish the relation between angular velocity and linear velocity. [3]

   (b) Derive the expressions for the tangential and normal acceleration. [7]

4. (a) A particle describes a catenary under a force which acts parallel to the axis. Find the law of force and the velocity at any point of the path. [5]

   (b) Show that a rectangular hyperbola can be described by a particle under a force parallel to an asymptote which varies as the cube of the distance from the other asymptote. [5]

5. (a) Prove that the central orbit is a plane curve. [3]

   (b) A boat, which is rowed with constant velocity u, starts from a point A on the bank of a river which flows with a constant velocity v and it points always towards a point B on the other bank exactly opposite to A. Find the equation of the path of the boat. If $u = v$, then show that the path is a parabola whose focus is B. [7]

6. (a) Define areal velocity. [3]

   (b) If a particle describes a circle of radius a under a force from a point at distance $c(< a)$ from the centre, then show that the force varies as

$$r\left(r^2+a^2-c^2\right)^{-3}.$$

[7]

Four Year Undergraduate Examination - 2024

Semester - III

Mathematics

Paper: SECMA - 03

(Number Theory)

Time: 3 Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer Question No. 1 and any five from the remaining questions

1. (a) Let a, b and n be positive integers with n odd. Prove that $(a+b) \mid (a^n + b^n)$.

   (b) Prove that the n^{th} prime number is strictly bigger than $2n-1$ for all integer $n \geq 5$.

   (c) Prove that $\gcd(21n+4, 14n+3) = 1$ for all integer $n \geq 1$.

   (d) Let m and n be positive integers such that $\frac{1}{m} + \frac{1}{n} \in \mathbb{Z}$. Prove that $m = n$.

   (e) State the law of quadratic reciprocity. [$5 \times 2 = 10$]

2. (a) Determine all prime numbers p such that $3p+1$ is a perfect square.

   (b) If both p and p^2+8 are prime numbers, then prove that p^3+4 is also a prime number.

   (c) Let $p_1, \ldots p_k$ be prime numbers, all bigger than 5, such that 6 divides $(p_1^2 + \ldots + p_k^2)$. Prove that 6 divides k. [3+3+4]

3. (a) For any integer $n \geq 1$, prove that the unit's digit in the decimal expansion of $n^2 - n + 7$ is either 3 or 7 or 9.

   (b) Let $n \geq 1$ be an integer such that both $2n+1$ and $3n+1$ are perfect squares. Prove that 40 divides n.

   (c) If $p \geq 5$ is a prime number, then prove that 13 divides $(10^{2p} - 10^p + 1)$. [3+4+3]

4. (a) Let $n \geq 1$ be an integer such that $\gcd(n, 35) = 1$. Prove that $n^{12} \equiv 1 \pmod{35}$.

   (b) Determine, with proof, if there exist positive integers m and n such that $n^2+n+1 = m^2$.

   (c) For any integer $n \geq 1$, prove that there exist n consecutive positive integers, none of which is a prime or a prime power. [4+3+3]

5. (a) Prove that an integer $n \geq 1$ is a perfect square if and only if $\tau(n)$ is odd.

   (b) For an integer $n \geq 3$, prove that $\displaystyle\sum_{k=1}^{n} \mu(k!) = 1$.

   (c) Determine, with proof, all prime numbers p such that $\tau(p^2+11) = 6$. [3+3+4]

6. (a) Prove that there exist infinitely many integers $n \geq 1$ for which $\phi(n)$ is a perfect square.

(b) If p and $2p+1$ are both odd prime numbers, then prove that $\phi(4p+2) = \phi(4p) + 2$.

(c) Let d and n be positive integers such that $d \mid n$. Prove that $\phi(d) \mid \phi(n)$.

(d) For a square-free integer $n \geq 1$, prove that $\tau(n^2) = n$ if and only if $n = 3$. [2+2+3+3]

7. (a) Let $p \equiv 1 \pmod 4$ a prime number and let g be a primitive root of p. Prove that $-g$ is also a primitive root of p.

(b) Let p be an odd prime number. Solve the congruence $X^{p-2} + \ldots + X + 1 \equiv 0 \pmod p$.

(c) Prove that the congruence $X^3 \equiv 3 \pmod{19}$ has no solutions. [4+3+3]

8. (a) Determine, with justification, if there exist integers x and y satisfying $y^2 = 43x^3 + 42$.

(b) Let p be a prime number. Prove that $p \mid (n^2-2)(n^2-3)(n^2-6)$ for some integer $n \geq 1$.

(c) Let $p = 2^k + 1$ be a prime number. If a is an integer with $1 \leq a \leq p-1$, then prove that $\left(\frac{a}{p}\right) = -1$ if and only if a is a a primitive root of p. [3+3+4]

9. (a) Determine all $x, y \in \mathbb{N}$ such that $x^2 - y! = 2001$.

(b) Let a, b and c be positive integers such that $a^2 + b^2 = c^2$. Determine, with justification, if 60 divides abc.

(c) Let $n \geq 2$ be an integer. Prove that $\sum_{d \mid n} \mu(d) = 0$. [3+3+4]