Use separate answer script for each unit

B. Sc. (Honours) Examination-2024

Semester-VI (CBCS)
Mathematics
Course: CCMA-13
(Analysis-VI)

Time: 3 Hours

Full Marks: 60

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Complex Analysis)
(Marks: 30)
Answer *any three* questions.

1. (a) If f is an analytic function on a domain D such that the argument of $f(z)$ is constant for all $z \in D$, then show that f must be constant.

   (b) Let $z_n = x_n + iy_n$ and $z_0 = x_0 + iy_0$. Prove that the necessary and sufficient condition that the sequence $\{z_n\}$ converges to z_0 is that $\{x_n\}$ converges to x_0 and $\{y_n\}$ converges to y_0. [4+6]

2. (a) Find an analytic function $f(z) = u + iv$ on $\mathbb{C}$ such that $f(0) = 0$ and $u(x, y) = x + e^{-x}\cos y$.

   (b) When is series $\sum_{n=1}^{\infty} z_n$ said to be conditionally convergent? Test the series $\sum_{n=1}^{\infty} \frac{n^2 2^n}{(1+i)^n}$ for convergence. [5+(1+4)]

3. (a) Evaluate $\int_\Gamma (\bar{z}dz + zd\bar{z})$ along the curve Γ defined by $z^2 + 2z\bar{z} + (\bar{z})^2 = (3+i)z + (3-i)\bar{z}$ from the point $z = 1+i$ to the point $z = 2-2i$.

   (b) Find all the Möbius transformations of the lower half-plane $\mathrm{Im}(z) \leq 0$ onto the unit disc $|w| \leq 1$. [5+5]

4. (a) Define the cross ratio of four complex numbers. Show that it is invariant under Möbius transformations. Hence find the Möbius transformation which maps $i, 2, -2$ onto $i, 1, -1$ respectively.

   (b) Evaluate $\oint_C \frac{zdz}{(z^2+25)(z-2i)}$, where C is the (positively oriented) circle $|z| = 3$. [(1+3+3)+3]

5. (a) State Liouville's theorem. Hence show that if f is an entire function such that $|f(z)| \leq |\sin z|$ $\forall\, z \in \mathbb{C}$, there is a constant c with $|c| = 1$ such that $f(z) = c\sin z$.

   (b) Let $P(z)$ be a polynomial of degree n and $a \in \mathbb{C} \setminus \{0\}$. Then show that there are exactly n points in $\mathbb{C}$ at which $P(z)$ assumes the value a. [(1+4)+5]

Unit-II (Metric Spaces)

(Marks: 30)

Answer *any three* questions.

1. (a) Define a metric space. For any four points x, y, z, w in a metric space (X, ρ), prove that $|\rho(x, y) - \rho(z, w)| \leq \rho(z, x) + \rho(w, y)$. [1+2]

   (b) Let $X = R^n$ and $x = (x_1, x_2, \ldots\ldots, x_n)$, $y = (y_1, y_2, \ldots\ldots, y_n)$ be in X. Show that (X, ρ) is a metric space where $\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i|$. [3]

   (c) Show that the space of all bounded sequences of real numbers with a suitable metric defined by you form a metric space. [4]

2. (a) Define an open set in a metric space. Let (X, ρ) be a metric space and $G \subset X$. Then show that G is an open set if and only if it is the union of open spheres. [1+3]

   (b) Show that union of finite number of open sets is open in a metric space. [3]

   (c) Define the closure $\bar{F}$ of a set $F \subset X$ in a metric space (X, ρ). Prove that the closure of a set F consisting of a single point is the set F. [1+2]

3. (a) Define a closed set in a metric space (X, ρ). Prove that the intersection of an arbitrary number of closed sets is a closed set. [1+2]

   (b) Define a Cauchy sequence in a metric space (X, ρ). Prove that if a sequence $\{x_n\}$ converges then it is a Cauchy sequence. [1+2]

   (c) Define a complete metric space. Prove that $\mathbb{R}^3$ is a complete metric space. [1+3]

4. (a) State and prove Cantor's intersection theorem. [1+5]

   (b) State and prove Baire's category theorem. [1+3]

5. (a) Examine if an open interval $(0, 2)$ with respect to usual metric of reals is an incomplete metric space. [2]

   (b) Define a contraction mapping over a metric space. Give an example of a contraction mapping having no fixed point. [1+2]

   (c) Show that a function $f : X \to Y$ is continuous if and only if $f^{-1}(G)$ is an open subset of X whenever G is an open subset of Y. [5]

B.Sc. (Honours) Examination-2024

Semester-VI

Mathematics

Course: CCMA - 14 [New]

(Algebra - IV)

Time: 3 Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any six*** questions.

1. (a) Let $\phi : G_1 \to G_2$ be a group homomorphism and let $x_1 \in G_1$. If $o(x_1)$ is finite, then prove that $o(\phi(x_1))$ divides $o(x_1)$.

   (b) Determine, with justification, the number of group homomorphisms from $\mathbb{Z}/8\mathbb{Z}$ to $\mathbb{Z}/5\mathbb{Z}$.

   (c) Let $\phi : G_1 \to G_2$ be a group homomorphism. Prove that ϕ is injective if and only if $\ker(\phi) = \{e_1\}$. [3+3+4]

2. (a) Let H be a subgroup of a group G. When do you say that "H is a characteristic subgroup of G"?

   (b) If H char G, then prove that $H \trianglelefteq G$.

   (c) Let H be a subgroup of a group G and let $g \in G$. Prove that $H \simeq gHg^{-1}$, where $gHg^{-1} = \{ghg^{-1} : h \in H\}$. [2+4+4]

3. (a) Let G' be the commutator subgroup of G. Prove that $G' \trianglelefteq G$ and G/G' is abelian.

   (b) Let $N \trianglelefteq G$ be such that $N \cap G' = \{e\}$. Prove that $N \subseteq Z(G)$. [6+4]

4. (a) Let R be a ring of characteristic p, where p is a prime number. Prove that the map $\phi : R \to R$ defined by $\phi(r) = r^p$ is a ring homomorphism.

   (b) Prove that the fields $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic.

   (c) Let $\phi : \mathbb{Q} \to \mathbb{Q}$ be a ring homomorphism. Prove that $\phi(x) = x$ for all $x \in \mathbb{Q}$. [4+3+3]

5. (a) Let R be a ring and let I_1 and I_2 be two ideals of R with $I_1 \subseteq I_2$. Prove that $R/I_1 \Big/ I_2/I_1 \simeq R/I_2$.

   (b) Prove that the ring $C[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous }\}$ is not an integral domain.

   (c) Let R be an integral domain with fraction field K. If S is an integral domain with $R \subseteq S \subseteq K$, then prove that $\mathbb{Q}(S) = K$. [4+3+3]

6. (a) Prove that every non-zero prime ideal in a Boolean ring is a maximal ideal.

   (b) Prove that $I = \{f \in C[0,1] : f(\frac{1}{2}) = 0\}$ is a maximal ideal in $C[0,1]$.

   (c) Give an example, with justification, of an ideal J of $C[0,1]$ that is not a prime ideal. [4+4+2]

7. (a) Let F be a field and let V be a vector space of dimension n over F. Let B be a fixed $n \times n$ matrix over F. Prove that $T : V \to V$ defined by $T(A) = AB - BA$ is a linear transformation.

(b) Let V be an n-dimensional vector space over a field F and let $T : V \to V$ be a linear operator such that $\mathrm{Im}(T) = \ker(T)$. Prove that n is an even integer. Moreover, find an example of such a linear operator for some vector space V.

(c) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V over a field F. Prove that the following two statements are equivalent.

(i) $\ker(T) \cap \mathrm{Im}(T) = \{0\}$.

(ii) If $T(T(\alpha)) = 0$ for some $\alpha \in V$, then $T(\alpha) = 0$.

[3+3+4]

8. (a) Let V and W be finite-dimensional vector spaces over a field F and let $T : V \to W$ be a linear transformation. Prove that $\ker(T^t) = (\mathrm{Im}(T))^0$.

(b) Let $A \in M_{n\times n}(\mathbb{R})$ be such that $\mathrm{trace}(A^t A) = 0$. Prove that $A = 0$.

(c) Let V be a finite-dimensional vector space over $\mathbb{R}$ and let $f, g \in V^*$. Assume that $h : V \to \mathbb{R}$ defined by $h(\alpha) = f(\alpha)g(\alpha)$ is also an element of V^*. Prove that either $f \equiv 0$ or $g \equiv 0$.

[3+3+4]

9. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator such that $T^2 \equiv 0$. Prove that either $T \equiv 0$ or there exists a basis of $\mathbb{R}^2$ relative to which the matrix of T is $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

(b) Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$. Prove that A is diagonalizable over $\mathbb{R}$.

(c) Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix} \in M_{4\times 4}(\mathbb{R})$. Find, with justification, conditions on a, b and c such that A is diagonalizable over $\mathbb{R}$.

[3+4+3]

B.Sc. (Honours) Examination, 2024
Semester-VI (CBCS)
Mathematics
Course: DSEMA-3A
(Computer Fundamentals)

Time: Two Hours　　　　Full Marks: 20

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. What do you mean by a flowchart? Using standard symbols, draw a flowchart to calculate and print the value of S, where $S = \sum_{J=1}^{100}\sum_{I=1}^{30} I \times J$. [1+4]

x4

2. (a) Name different types of variables with appropriate examples which are recognised by *FORTRAN* 77. [2]

(b) Write down the *FORTRAN* expression of the following mathematical expression: [2]

x4.

$$\sinh\left(\frac{a^3 + 3d^2 - \log_{10} d}{\tan(b^2 + c^2)e^{-|x^3y|}}\right) + \frac{x + n\cos^{-1} x}{\sqrt{2024 + x^3y^3}}$$

(c) Given the logical variables X, Y and Z, write a logical expression that is true if and only if X and Y are true and Z is false. [1]

3. (a) What values will be printed for the variables after execution of the following program: [3]

```
INTEGER D
LOGICAL B
COMPLEX A
CHARACTER*8 C
READ*, X, Y, Z, A, B
Y = Y * Y
Z = X + Y + Z
X = Y + Z
C = 'HEATWAVE'
B = .FALSE.
D = A + X
PRINT*, X, Y, Z, C, B, D
STOP
END
```

x2

The input data/line is given as 2.0, 3.0, 5.0, (−1., 3.0), .*TRUE*.
(You must mention all the necessary steps)

(b) Design an algorithm to compute the factorial of a positive integer n. [2]

x2

4. (a) Write a *FORTRAN* 77 program to arrange a sequence of n real numbers ($n \leq 2024$) in ascending order. [3]

(b) Write a *FORTRAN* 77 statement to carry out the following actions:
If $n = 1, 2, 8$, transfer control to statement 20, if $n = 3, 7$, transfer control to the statement 21, if $n = 4, 5, 6$ transfer control to the statement 22. [2]

[1 × 3]

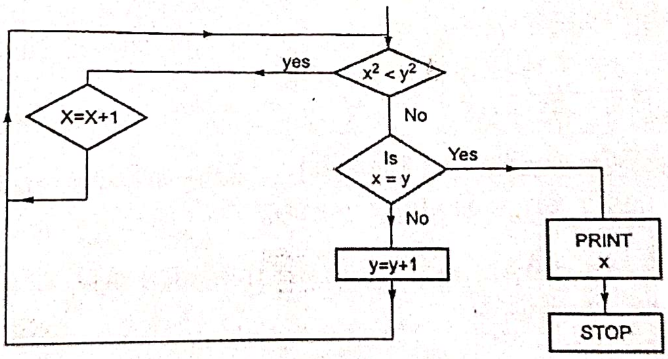
5. (a) Point out errors (if any) in the following. Justify your answer.
   (i) $GO\ TO(1, 2, 3, 4), VALUE$
   (ii) $READ*, X, Y + 3, Z, Z5$
   (iii) $TAN = TAN(X) * X * *0.5$

   (b) Rewrite the following $FORTRAN$ 77 segment using IF statements. [2]

```
DO 10 I = 1, 100
DO 20 J = 1, 25
WRITE(*, 1) I, J
20 CONTINUE
10 CONTINUE
```

6. (a) State (with examples) two important differences between $ARITHMETIC\ \ IF$ statement and $LOGICAL\ \ IF$ statement. [2]

   (b) Write the $FORTRAN$ 77 program segment for the following flowchart using $LOGICAL\ \ IF$ statement. [3]



B. Sc. (Honours) Examination-2024
Semester-VI (CBCS)
Mathematics
Paper: DSEMA-4
(Mathematical Modelling)

Time: 3 Hours

Full Marks: 60

Questions are of values as indicated in the margin.
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Group A (Physical System)
(Full Marks: 30)
Answer *any five* questions.

1. (a) Find the differential equation (DE) for radioactive decay. Solve the equation and explain it.
   (b) What do you mean by the half-life of a radioactive element? Show that the half-life of the radioactive element is $T_h = \frac{1}{k \ln 2}$, k is a positive constant. [3+3]

2. (a) Define the mixing problem in fluids and hence obtain the conservation law of mass for this case. Find the ordinary differential equation (ODE) for the mixing problem.
   (b) When does a partial differential equation (PDE) has well-defined solutions? Define Dirichlet and Neumann boundary conditions for a PDE. [4+2]

3. Let a product's price, denoted by $p(t)$, and its demanded and supplied quantities, denoted by Q_d and Q_s, respectively; Q_d and Q_s are the linear functions of $p(t)$. Obtain the first order ODE for the supplied and demanded model and hence solve it. Give a statement of the underlying assumptions. [6]

4. Derive the heat equation by using Fourier law and conservation of energy. [6]

5. (a) Derive the (1+1) dimensional wave equation.
   (b) What is meant by Carbon-dating? A fossilized bone is found to contain 0.1% of its original 14C. Find the age of the fossil. (Consider half-life $T_h = 5730$ years, $\ln 1000 = 6.91$) [3+3]

6. Write a short note on the followings:
   (a) Static model and dynamic model,
   (b) Linear model and nonlinear model,
   (c) Deterministic model and stochastic model. [2+2+2]

7. (a) State Newton's Law of cooling. Find the mathematical model and obtain the solution.
   (b) A horse shoe is heated to 100°C and then placed in a room to cool. The temperature of the room is 20°C. After 20 minutes, the temperature of the bar is 50°C. It is safe to handle at 30°C, how long must we wait to handle it? (Consider $\ln \frac{4}{9} = -0.81$, $\ln \frac{2}{9} = -1.504$) [3+3]

8. Find the DE for forced oscillations. What are meant by the transient solution and steady state solution? Obtain the amplitude and angular frequency of the transient and steady state solutions. Explain the condition under which resonance occurs. [6]

Group B (Biological System)
(Full Marks: 30)

Answer question number 1 and *any two* from the rest.

1. Answer *any two* questions.
(a) Consider the set of non-linear differential equations

$$\frac{dx}{dt} = 2x + 3y + 5x^2 - 10y^2,$$
$$\frac{dy}{dt} = 4x - 3y + 7xy.$$

Verify that $(0,0)$ is an equilibrium point. Show that the system is almost linear and discuss the type and stability of the equilibrium point $(0,0)$.

(b) Consider the following competing species model:

$$\frac{dx}{dt} = x(5 - y - x),$$
$$\frac{dy}{dt} = y(14 - 5x - 2y),$$
$$x(t=0) \geq 0, \quad y(t=0) \geq 0.$$

(i) Sketch the phase portrait using the method of nullclines.
(ii) What happens to the competing populations as time increases?

(c) What is a functional response? Write down the mathematical expressions and graphical representations of Holling type-I, type-II, type-III, and type-IV functional responses. [2 × 5=10]

2. The generalized Verhulst population model, with crowding effect of the form ($-\beta N^{\alpha}$) is given by

$$\frac{dN}{dt} = rN - \beta N^{\alpha} \ (\alpha > 1).$$

Obtain the steady-state solution and check for stability. Also, solve the model for $N(t)$ and predict the behaviour of the population for a long period of time. Find the point of inflexion, if any. [3+4+3]

3. Enlist the basic equations for a Lotka-Volterra predator-prey model. Carry out the stability analysis for the co-existence equilibrium point. Plot the graph for the predator-prey isocline through the vector-sum method to outline the dynamics of the four zones created by the isoclines. [2+5+3]

4. Define a reaction-diffusion system using an illustration.
Consider the following reaction-diffusion system:

$$\frac{\partial u(x,t)}{\partial t} = f(u) + D\frac{\partial^2 u(x,t)}{\partial x^2}.$$

(a) Explain the model mathematically as well as dimensionally.
(b) Write down the importance of diffusion parameter D on the system dynamics about the non-trivial equilibrium point.

[2+3+5]