Use separate answer script for each unit

B. Sc. (Honours) Examination-2024

Semester-IV (CBCS)
Mathematics
Paper: CC 8
(Analysis-IV and Differential Equations-II)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Analysis-IV)
(Full Marks: 30)
Answer *any three* questions.

1. (a) Prove that if $\lim_{(x,y)\to(\alpha,\beta)} f(x,y) = \lambda \neq 0$, then $f(x,y)$ has the sign of λ in some deleted neighbourhood of the point (α,β). [3]

   (b) Using $\epsilon - \delta$ definition, show that $\lim_{(x,y)\to(0,0)} \frac{xy^2(x^2-y^2)}{x^2+y^2} = 0$. [3]

   (c) If $f(x,y)$ is continuous at a point (α,β) then show that the function $f(\alpha,y)$ of the single variable y is continuous at $y=\beta$. Show by an example that the converse need not be true. [2+2]

2. (a) State and prove a sufficient condition for continuity of a function $f(x,y)$ at a point (α,β). [1+3]

   (b) When a function $f(x,y)$ is said to be differentiable at a point (α,β)?
   Prove that if $f(x,y)$ is differentiable at a point (α,β), then it is necessarily continuous at (α,β). [1+2]

   (c) Show that the function $f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}} & \text{for } (x,y)\neq(0,0) \\ 0 & \text{for } (x,y)=(0,0) \end{cases}$
   is not differentiable at $(0,0)$. [3]

3. (a) If $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x,y)\neq(0,0) \\ 0 & \text{for } (x,y)=(0,0), \end{cases}$
   then show that $f_{xy}(0,0) \neq f_{yx}(0,0)$. [4]

   (b) State and prove Young's theorem for equality of two mixed second order partial derivatives of a function $f(x,y)$ at a point (α,β). [1+5]

4. (a) State and prove Implicit function theorem. [2+4]

   (b) If y_1, y_2, y_3 are determined as functions of x_1, x_2, x_3 by the equations $f_r(x_1,x_2,x_3,y_1,y_2,y_3) = 0$; $r = 1,2,3$ then prove that
   $\frac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)} = -\frac{\partial(f_1,f_2,f_3)}{\partial(y_1,y_2,y_3)} \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$. [4]

5. (a) State and prove mean value theorem for a function of two variables. [1+3]

   (b) Show that for the function $f(x,y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$, $f(0,0)$ is neither a maximum value nor a minimum value. [3]

   (c) Evaluate $\int\int_E xy\,dx\,dy$, where E is the region bounded by x-axis, $x=1$ and $x^2 = 8y$. [3]

Unit-II (Differential Equations-II)
(Full Marks: 30)

Answer question number 1 and ***any two*** from the rest.

1. Answer ***any five*** questions.

(a) Are the two integrals $y_1 = \sin x + \cos x$ and $y_2 = \sin x - \cos x$ generate a set of fundamental solutions of the ordinary differential equation (ODE) $\frac{d^2y}{dx^2} + y = 0$? What is your conclusion?

(b) Write down the condition of exactness for the following linear ODE of order n:

$$P_0(x)\frac{d^ny}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_n(x)y = Q(x).$$

Hence, comment on the exactness of the ODE $x^5\frac{d^2y}{dx^2} + 3x^3\frac{dy}{dx} + (9 - 20x)x^2y = x^8 + x^4 + 1$.

(c) Given the solution $y_1(x) = \frac{\cos x}{x}$, find the second linearly independent solution $y_2(x)$ of the ODE

$$(x\sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y = 0, \quad x\sin x + \cos x \neq 0.$$

(d) Solve the following total differential equation by inspection method:

$$\frac{yz}{x^2+y^2}dx - \frac{xz}{x^2+y^2}dy - \tan^{-1}\left(\frac{y}{x}\right)dz = 0.$$

(e) Construct two first-order differential equations from the following ODE:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 2024\,x = 0$$

with $x = -46$ and $\frac{dx}{dt} = 44$ at $t = 0$.

(f) Consider the system of linear ODEs

$$\begin{aligned} \frac{dx}{dt} &= -y, \\ \frac{dy}{dt} &= x, \\ x(0) &= 2, \ y(0) = 3. \end{aligned}$$

Find the value of $x^2 + y^2$ at $t = \frac{\pi^c}{6}$.

(g) Determine $f(z)$ at which the following Pfaffian differential equation in three variables is integrable:

$$x^2dx + 3y^2dy + \{x^3 + 3y^3 + 3f(z)\}dz = 0.$$

(h) Locate and classify the regular and irregular singular points of the ODE

$$x^2(x-2)^2\frac{d^2y}{dx^2} + 9(x-2)\frac{dy}{dx} + (x+6)y = 0.$$

[5 × 2=10]

2. (a) Transform the second order linear ODE

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

by changing the independent variable from x to z into $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$ where $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{(\frac{dz}{dx})^2}$, $Q_1 = \frac{Q}{(\frac{dz}{dx})^2}$ and $R_1 = \frac{R}{(\frac{dz}{dx})^2}$.

(b) Solve the following second order linear ODE with variable coefficients:

$$\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - y\sin^2 x = \cos x - \cos^3 x.$$

(c) Comment on the number of independent arbitrary constants within the general solution of the following simultaneous equations:

$$\begin{aligned} (D-1)x + Dy &= 2t + 1, \\ (2D+1)x + 2Dy &= t, \end{aligned}$$

where $D \equiv \frac{d}{dt}$.

[4+4+2]

3. (a) How to find the general solution of the simultaneous differential equation of type II:

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}?$$

Hence find the general solution of the following simultaneous differential equation:

$$\frac{dx}{x+y-xy^2} = \frac{dy}{x^2y-x-y} = \frac{dz}{z(y^2-x^2)}.$$

(b) Solve the following system of two linear homogeneous differential equations with constant coefficients using eigenvalues and eigenvectors of the matrix:

$$\begin{aligned} \frac{dx}{dt} &= 4x - y, \\ \frac{dy}{dt} &= x + 2y, \\ x(t=0) &= 5, \;\; y(t=0) = 10. \end{aligned}$$

[(1+4)+5]

4. (a) Find the general solution of $[(x+2)D^2 - (2x+5)D + 2]y = (1+x)e^x$, $D^n \equiv \frac{d^n}{dx^n}$ by the method of operational factors.

(b) Use power series method to solve the following ODE about the ordinary point $x_0 = 0$:

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

[4+6]

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B. Sc. (Honours) Examination-2024

Semester-IV
Mathematics (Major)
Paper : CC-9 (Mechanics-I)
(Dynamics of a Particle and Statics)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I [Dynamics of a Particle (Marks: 30)]

Answer *any three* questions.

1. Define central force. A particle is let fall from rest from a point outside the earth at a distance b from the centre. Prove that the square of the velocity of the particle on reaching the centre is $ga\left(3-\frac{2a}{b}\right)$, where a is the radius of the earth and g is the value of gravity at its surface. [3+7]

2. (a) Obtain the expressions for tangential and normal accelerations of a particle moving in a plane curve. [5]
   (b) A particle describes a catenary under a force which acts parallel to the axis. To find the law of force and the velocity at any point of the path. [5]

3. (a) Obtain the pedal equation of central orbit. [5]
   (b) A particle describes an equiangular spiral $r = ae^{m\theta}$ with constant velocity. Find the components of velocity and of acceleration along the radius vector and perpendicular to it. [5]

4. (a) If a particle describes a circle of radius a under a force from a point at distance $c\ (< a)$ from the centre, show that the force varies as
$$r\left(r^2+a^2-c^2\right)^{-3}.$$
[5]
   (b) A particle subjected to the central acceleration $\left(\frac{\mu}{r^3}+f\right)$ is projected from an apse at a distance a with a velocity $\frac{\sqrt{\mu}}{a}$. Prove that at any subsequent time t, $r = a - \frac{1}{2}ft^2$. [5]

5. Find the equation of a trajectory when a particle is projected with a velocity u at an angle α to the horizon in a medium whose resistance is $mk\times$ (velocity). [10]

Unit-II [Statics (Marks: 30)]

Answer *any three* questions.

1. Define astatic equilibrium. Prove that the astatic centre is a fixed point. [3+7]

2. (a) A number of coplanar forces acting on a rigid body reduces to a single force only. Write down the equation of the line of action of their resultant (Define explicitly the parameters involved). [3]
   (b) Three forces P, Q, R act along the sides of the triangle formed by the lines $x+y=1, y-x=1, y=2$. Find the equation of the line of action and the magnitude of their resultant . [7]

3. State the principle of virtual work for a rigid body and hence deduce the conditions for equilibrium. [3+7]

4. (a) Two equal uniform rods AB and AC, each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that, if 2θ be the angle between them, then $b\sin^3\theta = a\cos\theta$. [5]
   (b) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that $\tan\phi = \frac{3}{8} + \tan\theta$. [5]

5. (a) Obtain the equations of the central axis of a given system of non-coplanar forces acting on a rigid body. [5]
   (b) Forces X, Y, Z act along the three straight lines $y=b, z=-c$; $z=c, x=-a$; $x=a, y=-b$ respectively; show that they will have a single resultant if
$$\frac{a}{X}+\frac{b}{Y}+\frac{c}{Z}=0.$$
[5]

B. Sc. (Honours) Examination-2024
Semester-IV (CBCS)
Mathematics
Course: CCMA 10
(Probability Theory)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer question no. 1 and ***any five*** from the rest.

1. Answer ***any five*** questions. $[5 \times 2 = 10]$
   (a) If A and B are two events such that $P(A) = P(B) = 1$, then show that $P(A+B) = P(AB) = 1$.
   (b) What is the probability of an odd sum when two dice are thrown?
   (c) If $F(x)$ denotes the distribution function of a random variable X, then prove that $F(\infty) = 1$.
   (d) Define Poisson process and give an example.
   (e) If X is a $\beta_2(l, m)$ variate, then show that $Y = 1/X$ is a $\beta_2(m, l)$ variate.
   (f) Define the moment generating function of a random variable X and find it for the binomial (n, p) distribution.
   (g) Prove that $E(X^2) \geq [E(X)]^2$.
   (h) If X is a non-negative random variable having mean m, prove by using Tchebycheff's inequality that, for any $\tau > 0$, $P(X \geq \tau m) \leq 1/\tau$.

2. (a) Using the axioms of mathematical probability, deduce the classical formula of probability. [3]
   (b) If $\{A_n\}$ is a monotonic non-decreasing sequence of events, then prove that [3]
   $$P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n).$$
   (c) Two urns contain respectively 3 white, 7 red, 15 black balls, and 10 white, 6 red, 9 black balls. One ball is drawn from each urn. Find the probability that both the balls are of the same colour. [4]

3. (a) Define stochastic independence of two events. Show that if two events A and B are independent, then A and $\overline{B}$ are independent. [2]
   (b) State and prove Bayes' theorem. [2+3]
   (c) The chance that a doctor will diagnose a certain disease correctly is 60%. The chance that a patient will die under his/her treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of the doctor who had the disease died. What is the chance that his/her disease was diagnosed correctly? [3]

4. (a) If $F(x)$ denotes the distribution function of a random variable X, then show that (i) $P(a < X < b) = F(b-0) - F(a)$ and (ii) $P(a \leq X \leq b) = F(b) - F(a-0)$. [3]
   (b) Find the value of the constant k such that
   $$f(x) = \begin{cases} kx(1-x) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$
   is a probability density function corresponding to a random variable X. Find the distribution function of X and compute $P(X > 1/2)$. [1+2+1]
   (c) A point P is taken at random on a line segment AB of length $2a$. Find the probability that the area of the rectangle $AP.PB$ will exceed $a^2/2$. [3]

5. (a) The joint probability density function of the random variables X and Y is given by

$$f(x,y) = \begin{cases} k(3x+y) & 1 < x < 3,\ 0 < y < 2, \\ 0 & \text{elsewhere.} \end{cases}$$

Find the value of the constant k and calculate $P(X+Y<2)$. Examine if X and Y are independent. [1+2+2]

(b) If $X_1,\ X_2, ..., X_n$ are mutually independent random variables each uniformly distributed over $(0,1)$, prove that the probability density function of $X = X_1 X_2 \cdots X_n$ is

$$\frac{1}{(n-1)!}\left[\log\left(\frac{1}{x}\right)\right]^{n-1}, \quad (0 < x < 1).$$

Hence deduce the probability density function of the geometric mean $(X_1 X_2 \cdots X_n)^{1/n}$. [3+2]

6. (a) If X is a continuous random variable having spectrum $-\infty < x < \infty$ and distribution function $F(x)$, then show that

$$E(X) = \int_0^\infty G(x)dx,$$

provided $xG(x) \to 0$ as $x \to \infty$, where $G(x) = 1 - F(x) - F(-x)$. [4]

(b) Show that the characteristic function for the Pascal distribution, given by,

$$f_i = (1-\lambda)\lambda^i, \quad i = 0, 1, 2, ...,$$

is $\left[1 - \mu\left(e^{it} - 1\right)\right]^{-1}$, where $\mu = \lambda/(1-\lambda) > 0$. [3]

(c) Find the mode of a binomial $(4, 1/4)$ variate. [3]

7. (a) Define correlation coefficient $\rho(X,Y)$ between two random variables X and Y. If X and Y are uncorrelated, find the correlation coefficient between the linear combinations $a_1X + b_1Y$ and $a_2X + b_2Y$. [1+2]

(b) The probability density function of a continuous bivariate distribution is given by

$$f(x,y) = \begin{cases} x+y & 0 < x < 1,\ 0 < y < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Find the regression curves for the means of X and Y. [2+2]

(c) If, for any pair of linearly dependent random variables X and Y, we set $U = X\cos\alpha + Y\sin\alpha$ and $V = -X\sin\alpha + Y\cos\alpha$, then prove that V will be constant (i.e., it has a one-point distribution) if $\tan\alpha = \rho\sigma_y/\sigma_x$. Here, α is a constant and σ_x (σ_y) is the standard deviation of X (Y). [3]

8. (a) When is a sequence of random variables $\{X_n\}$ said to be asymptotically normal? State the Central limit theorem for the case of equal components and show that it implies the law of large numbers. [1+2+3]

(b) State the limit theorem for characteristic functions and use it to show that the Poisson distribution is a limit of the Binomial distribution. [2+2]

9. (a) Define convergence in probability for a sequence of random variables $\{X_n\}$ and state the law of large numbers for the case of equal components. [1+2]

(b) If f denotes the frequency ratio of successes in a Bernoullian sequence of n trials with probability of success p, then prove that [3]

$$f \xrightarrow{\text{in } p} p \quad \text{as } n \to \infty.$$

(c) Verify Tchebycheff's inequality for the probability distribution of a random variable X, given by, [4]

$$P(X=-1) = P(X=1) = \frac{1}{8}, \ P(X=0) = \frac{3}{4}.$$

B.Sc.(Honours) Examination-2024

Semester-IV

Mathematics

Course: SECMA-2

(Tensor Calculus)

Time: Two Hours **Full Marks: 25**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meaning.

Answer ***any five*** questions.

1. What do you mean by scalar, vector, and tensor fields? Define covariant, contravariant, and mixed tensors, their order, and rank. Transform the covariant components of a tensor field $(2x+y, xyz, (x+y)^2z)$ in Cartesian coordinates to components in spherical polar coordinates. [5]

2. Define inner multiplication, contraction, and quotient law of tensor fields. If A_l^{pq} and B_{rst}^{ij} are two tensor fields, examine which one among their sum, difference $A_l^{pq} \pm B_{rst}^{ij}$, and the product $A_l^{pq} B_{rst}^{ij}$ is a tensor field. Justify your answer. Prove that the contraction of $B_i C^j$ is a scalar field. [5]

3. If Φ be a scalar field and A_p be a covariant tensor field, examine which of the $\frac{\partial \Phi}{\partial x^l}$, $\frac{\partial A_p}{\partial x^q}$ is a tensor field? [2.5+2.5]

4. A field variable $A(i,j,k)$ is such that in the coordinate system x^p, $A(i,j,k)B_r^{pq} = C_i^q$ where B_r^{pq} is an arbitrary tensor field and C_i^q is a mixed (1,1) tensor field. Examine whether $A(i,j,k)$ is a tensor field. [5]

5. Define symmetric and skew-symmetric tensors. Prove that every tensor field can be expressed as the sum of symmetric and skew-symmetric tensors in a pair of covariant or contravariant indices. If $\Phi = a_{pq}A^pA^q$, show that Φ can be expressed as $\Phi = b_{ij}A^iA^j$, where b_{ij} is a symmetric tensor field. [5]

6. Define line element on a curve in $\mathbb{R}^n$ in terms of inner product of tensor fields. Examine the nature of the tensor field involved there. Define associated tensors corresponding to those tensor fields. Give the interpretation of these tensors in $\mathbb{R}^3$. Find the expressions for metric tensors on the spherical and cylindrical surfaces of unit radii. [5]

7. Define Christoffel's symbols of first and second kind. Evaluate the expressions for Christoffel's first- and second-kind symbols in orthogonal curvilinear coordinates in terms of partial derivatives of metric tensors. Hence obtain their expressions for cartesian coordinate systems. [5]

8. Define covariant derivatives of tensor fields. Is there any difference between partial and covariant derivatives of a tensor field? When are they same if your answer is affirmative? Prove that the covariant double derivative of a covariant vector field is not commutative, in general. [5]