M. Sc. Examination-2025

Semester-IV
Mathematics
Course: MMC-41(New)
(Differential Geometry and Manifold Theory)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. Define chart, atlas, maximal atlas and manifold ($\mathcal{M}$) on a second countable Hausdorff topological space. Introduce tangent vector at a point $p \in \mathcal{M}$, tangent space $T_p\mathcal{M}$ at a point $p \in \mathcal{M}$, tangent bundle $T\mathcal{M}$ on $\mathcal{M}$ and a vector field X on $\mathcal{M}$. Define metric tensor g_{ij} on $\mathcal{M}$ in terms of inner product on the tangent space. Obtain expressions for g_{ij} on $\mathcal{M} = S^2 = (\sin u \cos v, \sin u \sin v, \cos u)$. [4+4+1+1]

2. Introduce differential form, their wedge product, exterior derivatives and the Hodge $*$-map between the space of forms. Establish the formula relating exterior derivative of wedge product of two forms. Derive expressions for Laplacian (∇^2) operator as the combination of exterior derivative and Hodge-map in the manifold $\mathcal{M} = \mathbb{R}^3$. Hence or otherwise, derive expressions for Laplacian (∇^2 or $\triangle$) operator in an arbitrary curvilinear coordinate in $\mathbb{R}^3$. [4+2+2+2]

3. Define Lie group and Lie group of transformations. Find rules of binary composition and inversion for the set $\mathcal{S}(a, b, c) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} ; a, b, c \in \mathbb{R} \right\}$ to be a Lie group. Find infinitesimal generators for the Lie group of transformations $T_{(\epsilon_1,\epsilon_2,\epsilon_3)} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x^{*1}, x^{*2}) = T_{(\epsilon_1,\epsilon_2,\epsilon_3)}(x^1, x^2) = (e^{\epsilon_1}x^1 + \epsilon_2,\ e^{3\epsilon_1}x^2 + \epsilon_3)$. [2+2+3+3]

4. Introduce the frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$. Prove that derivative of tangent of a curve on some surface in $\mathbb{R}^3$ lies in the plane of $\mathbf{g}$ and $\mathbf{n}$. Define normal curvature, geodesic curvature and the geodesic torsion of a curve on a surface. Find coefficients of second fundamental form in terms of first and second derivatives of the position vector $\mathbf{r}(u, v)$ of the surface. Obtain coefficients of second fundamental form and normal curvature to a spherical surface of radius a [1+2+3+2+2]

5. Derive expressions for second derivatives of $\mathbf{r}(u, v)$ with respect to the parameters u, v in terms of Christoffel symbols of the second kind. Establish the formula $\Gamma_{ijk} = \mathbf{r}_{ij} \cdot \mathbf{r}_k = \frac{1}{2}(g_{ik};j + g_{jk};i - g_{ij};k)$; $i, j, k \in \{u, v\}$. Examine whether the following relations

$$\Gamma^u_{uu} = \frac{E_u}{2E},\ \Gamma^u_{uv} = \Gamma^u_{vu} = \frac{E_v}{2E},\ \Gamma^v_{vu} = \Gamma^v_{uv} = \frac{G_u}{2G},\ \Gamma^u_{vv} = -\frac{G_u}{2E},\ \Gamma^v_{uu} = -\frac{E_v}{2G}, \Gamma^v_{vv} = \frac{G_v}{2G},$$

among $\Gamma^i_{jk}(u, v)$ and the coefficients E, F, G of first fundamental form on a surface in $\mathbb{R}^3$ are valid. [2+3+5]

6. Define the Riemann-Christoffel curvature and the Ricci tensor. Establish that both tensors reflect the intrinsic property of a surface $\mathcal{S}$ in $\mathbb{R}^3$. Prove that Riemann-Christoffel curvature is antisymmetric in some indices that you must determine. Prove the Gauss theorem Egregium, i.e., Gauss curvature of a surface in $\mathbb{R}^3$ is an intrinsic property. [2+3+2+3]

M.Sc. Examination - 2025
Semester-IV
Mathematics
Paper: MMC-42
(Operations Research)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin
Notations and symbols have their usual meaning

Answer *any four* questions

1. Comment on 'Revised Simplex method is computationally efficient than Simplex method in solving LPPs'. Use Revised Simplex method to solve the LPP

$$\begin{aligned} \text{Maximize} \quad & Z = 2x_1 + x_2 \\ \text{subject to} \quad & 3x_1 + 4x_2 \leq 6 \\ & 6x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{aligned}$$

[3+7]

2. What is Integer programming? Justify the need for developing an efficient procedure for optimal integer solution. Solve the LPP

$$\begin{aligned} \text{Maximize} \quad & Z = 2x_1 + 3x_2 \\ \text{subject to} \quad & 6x_1 + 5x_2 \leq 25 \\ & x_1 + 3x_2 \leq 10 \\ & x_1, x_2 \text{ are non-negative integers.} \end{aligned}$$

[2+8]

3. (a) Solve graphically the NLPP:

$$\begin{aligned} \text{Maximize} \quad & Z = 2x_1 + 3x_2 \\ \text{subject to} \quad & x_1 x_2 \leq 8 \\ & x_1^2 + x_2^2 \leq 20 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(b) State Karush-Kuhn-Tucker condition for the solution of a convex programming problem and use it to

$$\begin{aligned} \text{Maximize} \quad & Z = x_1^2 - 2x_1 + x_2^2 + 1 \\ \text{subject to} \quad & x_1 + x_2 \leq 0 \\ & x_1^2 - 4 \leq 0. \end{aligned}$$

[5+5]

4. Explain Dynamic programming. Formulate Dynamic programming technique to solve a linear programming problem. Use it to

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 5x_2 \\ \text{subject to} \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0. \end{aligned}$$

[2+8]

5. (a) Determine the optimum quantity for each run for a single type of product and the optimum interval between successive runs where shortages are allowed.

(b) The demand for an item is 18000 units per year and the company can produce the item at a rate of 3000 units per month. The cost of one set up is Rs. 500 and the holding cost of one unit per month is 15 paise. The shortage cost of one unit is Rs. 20 per month. Determine the optimum manufacturing quantity and the number of shortages. Also determine the manufacturing time and the time between set ups. [6+4]

6. (a) For the Queue model (M/M/1: N/FCFS), find the probability of n customers in the system.

(b) Trains arrive at a railway yard every 15 minutes and the service time is 33 minutes. If the line capacity of the yard is limited to 4 trains, find
(i) the average number of trains in the system;
(ii) the average time a train spends in the system. [6+(2+2)]

Use separate answer script for each unit

M. Sc. Examination-2025

Semester-IV
Mathematics
Elective Course: MME-41 (Applied Stream)
(Electromagetic Theory and Programming in Matlab)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 20)
(Electromagnetic Theory)

Answer *any two* questions.

1. (a) State Coulomb's law in electrostatics for two point charges. [2]

   (b) Define an electric field. A long cylinder caries a charge density which is proportional to the distance from the axis, i.e., $\rho = ks$, where k is a constant. Find the electric field inside this cylinder. [1+3]

   (c) For a point charge configuration in a spatially uniform electric field, define an electric dipole. Calculate the potential energy of an electric dipole placed in an external electric field. [1+3]

2. (a) Derive the equation of continuity for a steady state current of electric charges. What is Ohm's law? Use it to show that for a steady state current, the continuity equation reduces to the Laplace equation for the electrostatic potential. [2+2+1]

   (b) Define the magnetic force acting on a charged particle of charge q. State and explain the Amperé's circuital law. What are the shortcomings of this law? [1+2+2]

3. (a) Show that for a continuous charge distribution in an electric field $\mathbf{E}$, the electrostatic energy density is given by $\mathcal{E} = \frac{1}{2}\varepsilon_0|\mathbf{E}|^2$. [5]

   (b) State and prove the Poynting's theorem. What is Poynting's vector? [1+3+1]

Unit-II (Full Marks: 20)
(Programming in Matlab)

Answer question no. 1 and *any one* from the rest.

1. Choose the correct alternative (*any five*).

   (a) What does MATLAB stand for? (i) Math Laboratory (ii) Matrix Laboratory (iii) Mathworks (iv) Mathematics Lab. [2]

   (b) Which MATLAB command is used to clear all data and variables stored in memory? (i) clear (ii) clc (iii) delete (iv) close. [2]

   (c) What is the output of $C = A. * B$ when $A = [1\ 0\ 2]$; $B = [3\ 0\ 7]$? (i) [2 0 21] (ii) [3 0 14] (iii) [14 0 3] (iv) [7 0 3]. [2]

   (d) The output of A=['Bat"Ball'] is (i) 'Bat'Ball' (ii) Bat Ball (iii) BatBall (iv) Bat & Ball. [2]

   (e) Index of an array in MATLAB starts with (i) 0 (ii) 1 (iii) Depends on the class of array (iv) Unknown. [2]

(f) Which type of analysis is not done by MATLAB software? (i) Numerical computing (ii) Algebraic solutions (iii) Plant planning analysis (iv) Dynamic system simulations. [2]

(g) Which of the following correctly defines x, y, and z as symbols? (i) sym (x,y,z) (ii) syms x y z (iii) syms x, y, z (iv) sym x, y, z. [2]

(h) Keys combination used to stop execution of a command in MATLAB. (i) ctrl+c (ii) ctrl+s (iii) ctrl+b (iv) ctrl+enter. [2]

2. (a) If $A = [1\ 1\ 1; 2\ 2\ 2; 3\ 3\ 3]$ is a matrix, then write the outputs of (i) $A(3,2)$, (ii) $A(2:3,3)$, (iii) $A(2,:)$, and (iv) $A(:,3)$. [6]

(b) Write the Matlab command to find the roots of a polynomial equation $2x^4 - 3x^2 + 5 = 0$. [2]

(c) Write a MATLAB program to plot the function $f(x) = x^3 e^{-2x^2} \cos x$, $-1 \leq x \leq 1$. [2]

3. Write a MATLAB program to solve the coupled nonlinear equations: $x = y + \cos(x + y)$, $y = x + \sin(xy)$. [10]

M.Sc. Examination-2025

Semester-IV

Mathematics

Optional Course: MMO 41 (A8)

(Mathematical Pharmacology-II)

Time: Three Hours Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Write down the mass balance equations for a two-compartment pharmacokinetics model with a first-order absorption alongwith a first-order elimination kinetics from the central and peripheral compartments. Obtain the transient drug concentration in each compartment. [3+7]

2. What is meant by an ideal stirred-tank model and an ideal plug flow model? Assuming flow-limited conditions, obtain the unsteady concentration for microvascular mixing in tissue compartments. [3+7]

3. Using appropriate initial and boundary conditions, obtain the time-dependent concentration of drug within a planar membrane. [10]

4. Define homogeneous and heterogeneous chemical reactions. Obtain the concentration in case of diffusion and chemical reaction inside a spherical porous catalyst (assuming appropriate boundary conditions). [10]

5. Discuss the pharmacokinetics of T-20 after intravenous administration and obtain the steady state solution for plasma concentration. [3+7]

6. Considering a reversible homogeneous reaction on solutes A and B in a stagnant liquid film, obtain the nondimensional concentrations of A and B. [10]

M. Sc. Examination-2025

Semester-IV

Mathematics

MMO 41 : Optional Paper

P 9 : Rings and Modules-II

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Every ring R contains unity and every module is unitary.

Answer ***any four*** questions.

1. (a) Let M be a left R-module. Prove that there is unique R-isomorphism $\phi : R \otimes M \longrightarrow M$ such that $\phi(r \otimes m) = rm$. [3]
   (b) Let U and V be two vector spaces with $\dim U = m$ and $\dim V = n$. Show that $\dim(U \otimes V) = mn$. [4]
   (c) Let R be a ring. Prove that R is regular if and only if every left R-module is flat. [3]

2. (a) Let $J(R)$ be the intersection of all maximal left ideals of R. Show that $J(R) = \{a \in R \mid 1 - ra \text{ is left invertible for every } r \in R\}$ [4]
   (b) Let R be a ring. Show that $J(R)$ is an ideal of R. Also show that $J(R/J(R)) = 0$. [3]
   (c) Let M be a left R-module. Show that $Rad_J(M) = \cap_f \ker f$ where each f is an R-morphism from M to a simple R-module. Hence or otherwise show that for every R-morphism $f : M \longrightarrow N$, $f(Rad_J(M)) \subseteq Rad_J(N)$. [3]

3. (a) Let M be a semisimple left R-module. Show that every submodule and homomorphic image of M is semisimple. [4]
   (b) If L is a minimal left ideal of a ring R and M is a simple left R-module then show that either $L \simeq M$ or $LM = 0$. Hence or otherwise show that for every non-isomorphic minimal left ideals L and K of R, $LK = 0$. [3]
   (c) Let R be a left semisimple ring. Show that R is isomorphic to a direct product of matrix rings over division rings. [3]

4. (a) Let R be a commutative ring with more than one element. If R is subdirectly irreducible, then show that R is a field. [4]
   (b) Show that every Boolean ring is a subdirect product of isomorphic copies of $\mathbb{Z}_2$. [3]
   (c) Show that $J(\mathbb{Z}) = 0$. Hence or otherwise show that $\mathbb{Z}$ is a subdirect product of finite fields. [3]

5. (a) Let M be a left R-module. If every submodule of M is finitely generated, then show that M is noetherian. [3]
   (b) Let M be a left R-module and N be a submodule of M. If both N and M/N are artinian, then show that M is artinian. [4]
   (c) Let M be a left R-module of finite height and N be a submodule of M. Prove that $h(N) + h(M/N) = h(M)$. [3]

6. (a) Give an example of a noetherian module. [2]
   (b) Find $J(\mathbb{Z}_{100})$. [2]
   (c) Show that every vector space is semisimple. [2]
   (d) Is $\mathbb{Q}$ a semisimple $\mathbb{Z}$-module? [2]
   (e) Show that $\mathbb{Z}_{50} \otimes \mathbb{Q} = 0$. [2]

Use separate answer script for each unit

M.Sc. Examination-2025

Semester-IV
Mathematics
MME-41
(Galois Theory -II and Algebraic Topology)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit I: Galois Theory II (Full Marks: 20)

Answer ***any four*** questions.

1. (a) Let F/K be a finite extension. Show that it is a normal extension if and only if F is a splitting field of a polynomial over K.

   (b) Give an example of a finite field extension that is not normal. [3+2]

2. (a) Let $f(x) \in K[x]$ be a nonconstant polynomial. If F_1 and F_2 be two splitting fields of $f(x)$ over K, show that $G(F_1/K) \simeq G(F_2/K)$.

   (b) If $[F : K] = 2$, show that F/K is a normal extension. [3+2]

3. (a) Let F be a field of order p^n. Show that $G(F/\mathbb{Z}_p) \simeq \mathbb{Z}_n$.

   (b) Let F/K be a finite field extension and H a subgroup of $G(F/K)$. Show that $H = G(F/F_H)$. [3+2]

4. (a) Define constructible numbers. Show that the set of all constructible numbers is a field.

   (b) Prove that $\sqrt{5}$ is constructible. [3+2]

5. (a) Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be n automorphisms of a field F. If $a_1, a_2, \cdots, a_n \in F$ are such that

$$a_1\sigma_1(x) + a_2\sigma_2(x) + \cdots + a_n\sigma_n(x) = 0, \quad \text{for all } x \in F,$$

   then show that $a_1 = a_2 = \cdots = a_n = 0$.

   (b) Show that the angle 60° can not be trisected by ruler and compass only. [3+2]

6. (a) Let F be the splitting field of a separable polynomial $f(x)$ over K. Show that $|G(F/K)| = [F : K]$.

   (b) Show that $|G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})| \neq [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$. [3+2]

Unit-II : Algebraic Topology (Full Marks: 20)

Answer any four questions

1. (a) Let $f : S^1 \to \mathbb{R}$ be a continuous map. Prove that there exists an element $x \in S^1$ such that $f(x) = f(-x)$. [2]

   (b) Is $GL_n(\mathbb{R})$ a path-connected subset of $\mathbb{R}^{n^2}$? Is it a compact subset of $\mathbb{R}^{n^2}$? Justify your answers. [3]

2. (a) Let G be a connected topological group. Prove that any open set containing the identity element of G is a generating set of G. [3]

   (b) Let G be a topological group with the identity element e. Let U be an open subset of G containing e. Prove that there exists an open set $V \subseteq G$ containing e such that $VV^{-1} \subseteq U$. [2]

3. (a) Let X be a path-connected topological space and let x_0 and x_1 be two distinct points in X. Prove that $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$. [3]

   (b) In a topological group $(G, \cdot)$ with the identity element e, prove that the operation $[f] \otimes [g] = [f \otimes g]$, where $(f \otimes g)(s) = f(s) \cdot g(s)$, defined on $\pi_1(G, e)$ is well-defined. [2]

4. (a) When is a map $p : E \to B$ said to be a covering map? Give an example of a map which is not a covering map. Justify your answer. [3]

   (b) Let $p : E \to B$ be a covering map and let $b \in B$. Prove that $p^{-1}(\{b\})$ is a discrete subset of E. [2]

5. (a) Prove that a covering map is an open map. [2]

   (b) Let $p : E \to B$ be a covering map. If B is Hausdorff, then prove that E is also Hausdorff. [3]

6. (a) Define lifting correspondence for a covering map $p : E \to B$. If E is simply connected, then prove that the lifting correspondence is bijective. [3]

   (b) Using lifting correspondence, prove that the fundamental group of a circle is isomorphic to $(\mathbb{Z}, +)$. [2]

M.Sc. Examination-2025

Semester-IV

Mathematics

Optional Course: MMO 41 (A8)

(Mathematical Pharmacology-II)

Time: Three Hours Full Marks: 40

Questions are of value as indicated in the margin.

Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Write down the mass balance equations for a two-compartment pharmacokinetics model with a first-order absorption alongwith a first-order elimination kinetics from the central and peripheral compartments. Obtain the transient drug concentration in each compartment. [3+7]

2. What is meant by an ideal stirred-tank model and an ideal plug flow model? Assuming flow-limited conditions, obtain the unsteady concentration for microvascular mixing in tissue compartments. [3+7]

3. Using appropriate initial and boundary conditions, obtain the time-dependent concentration of drug within a planar membrane. [10]

4. Define homogeneous and heterogeneous chemical reactions. Obtain the concentration in case of diffusion and chemical reaction inside a spherical porous catalyst (assuming appropriate boundary conditions). [10]

5. Discuss the pharmacokinetics of T-20 after intravenous administration and obtain the steady state solution for plasma concentration. [3+7]

6. Considering a reversible homogeneous reaction on solutes A and B in a stagnant liquid film, obtain the nondimensional concentrations of A and B. [10]

M. Sc. Examination-2025

Semester-IV

Mathematics

MMO-41 (P5)

(Algebraic Coding Theory-II)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Show that a nonempty subset C of $\mathbb{F}_q^n$ is a cyclic code if and only if $\pi(C)$ is an ideal of $\mathbb{F}_q[x]/(x^n-1)$, where $\pi : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q[x]/(x^n-1)$, is a homomorphism defined by $\pi(a_0, a_1, \dots, a_{n-1}) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}\ mod(x^n-1)$. [5+5]

2. (a) Determine whether the polynomial $g(x) = 1+x+x^2+x^3+x^4$ is a generator polynomial of a cyclic code of the given length 7. [3]

   (b) How many binary cyclic codes of length 6 are there? Construct a $[6, 4]$–binary cyclic code. [4+3]

3. (a) Let $g(x) = g_0 + g_1x + \cdots + g_{n-k}x^{n-k}$ be the generator polynomial of a cyclic code $C \subseteq \mathbb{F}_q^n$ with $\deg(g(x)) = n-k$. Find the corresponding generator matrix for the code C. [6]

   (b) Find the parity-check matrix of the binary $[7, 4]$-cyclic code generated by $g(x) = 1 + x^2 + x^3$. [4]

4. Show that a BCH code with designed distance δ has minimum distance at least δ. [10]

5. Determine the generator polynomials of the narrow-sense binary BCH codes of length 15. [10]

6. Consider the 8-ary Reed–Solomon code C generated by

$$g(x) = \prod_{i=1}^{6}(x-\alpha^i) = \sum_{i=0}^{6} x^i,$$

where α is a root of the primitive polynomial $1 + x + x^3$ over $\mathbb{F}_2$. Then show that $C = \{a(1, 1, 1, 1, 1, 1, 1) : a \in \mathbb{F}_8\}$ is a $[7, 1, 7]$-MDS code. Hence show that $\varphi^*(C)$, is a binary $[21, 3, 7]$-linear code, where $\phi : \mathbb{F}_{2^m} \to \mathbb{F}_2^m$ is a homomorphism defined by $\phi(u_1\alpha_1 + u_2\alpha_2 + \cdots + u_m\alpha_m) = (u_1, u_2, \dots, u_m)$ for basis $\alpha_1, \alpha_2, \dots, \alpha_m$ of $\mathbb{F}_{2^m}$. [3+7]

M.Sc. Examination-2025

Semester-IV

Mathematics

Paper: MMO 41 (P01)

(Advanced Complex Analysis-II)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Let $f(z)$ be meromorphic in $|z| < \infty$ and $a_1, a_2, a_3, \ldots\ldots$ be the poles of $f(z)$, where $0 < |a_1| \le |a_2| \le |a_3| \ldots\ldots$. Suppose that there is a sequence of closed contours C_n such that C_n includes $a_1, a_2, \ldots. a_n$ but no other poles and R_n is the minimum distance of C_n from the origin, which tends to infinity with n, while L_n, the length of C_n is $O(R_n)$. Now if $|f(z)| = o(R_n)$ on C_n and residue at a_m be b_m, then show that $f(z) = f(0) + \sum_{n=1}^{\infty} b_n(\frac{1}{z-a_n} + \frac{1}{a_n})$, for all values of z except the poles. [6]

   (b) Let $f(z)$ be meromorphic in $|z| < R$ and $f(0) \neq 0$. If it has zeros at $a_1, a_2, \ldots.. a_m$ and poles at $b_1, b_2, \ldots.. b_n$ with moduli not exceeding $r(0 < r < R)$ then show that $log\{|\frac{b_1.b_2.\ldots\ldots b_n}{a_1.a_2.\ldots\ldots a_m} f(0)|r^{m-n}\} = \frac{1}{2\pi}\int_0^{2\pi} log|f(re^{i\theta})|d\theta$. [4]

2. (a) Define Nevanlinna's characteristic function $T(r, f)$. Find $T(r, f)$ when $f(z) = e^{z^4}$. [2+2]

   (b) Show that for any a, $|T(r, f) - T(r, f - a)| \le log^+|a| + log2$. [2]

   (c) Using Cartan's identity prove that $\frac{1}{2\pi}\int_0^{2\pi} m(r, e^{i\theta})d\theta \le log2$. [4]

3. (a) State and prove Nevanlinna's first fundamental theorem. [1+3]

   (b) Prove that for any $'a'$, $N(r, a)$ is a convex function of $logr$. [2]

   (c) Define the order of a non-constant meromorphic function $f(z)$. Find the order of $f(z) = e^{-z}$. [1+3]

4. (a) If ρ_1 and ρ_2 are the orders of the meromorphic functions f_1 and f_2 respectively, then show that the order of $\frac{f_1}{f_2} \le max\{\rho_1, \rho_2\}$. [3]

   (b) If $f(z)$ be a non-constant meromorphic function in the complex plane and $'a'$ be any finite complex number, then define $\delta(a, f)$ and also find $\delta(0, e^{3z})$. [1+2]

   (c) State and prove Nevanlinna's theorem on deficient values. [1+3]

5. (a) Let $S(r)$ be a real valued and non negative increasing function for $0 < r_0 < r < \infty$. If $S(r)$ is of order $\rho(0 < \rho < \infty)$ and has convergence class, then prove that $S(r)$ has minimal type. Does the converse hold ? Support your answer. [2+3]

   (b) Give an example to show that there exist functions f and g share four values IM but $f \neq g$. [2]

   (c) If f and g are two non-constant polynomials share $'1'$ CM and $f(0) = g(0) \neq 1$, then show that $f \equiv g$. [3]

6. (a) State and prove Milloux theorem. [2+5]

   (b) Define an elliptic function. If $f(z)$ is an elliptic function then show that $f'(z)$ is also so. [1+2]

M. Sc. Examination-2025

Semester-IV

Mathematics

Course: MMO-41 (P-2)

(Advanced Functional Analysis-II)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any four questions

1. (a) Prove that every closed convex subset of a Hilbert space has a unique member of smallest norm. [4]

   (b) Show that a Hilbert space H contains a complete orthonormal sequence if and only if H is separable. [4]

   (c) Give an example (with justification) of a complete orthonormal sequence in a Hilbert space. [2]

2. (a) Prove that for every bounded linear operator $T : H \to H$ where H is a Hilbert space, its adjoint operator T^* exists which is unique and satisfying $||T^*|| = ||T||$. [4]

   (b) Let f be a sesquilinear form defined on a Hilbert space and $\hat{f}$ be the associated quadratic form of f. Prove that f is bounded iff $\hat{f}$ is bunded and moreover $||\hat{f}|| \leq ||f|| \leq 2||\hat{f}||$. [3]

   (c) Show that the collection of all self-adjoint operators defined on a real Hilbert space H form a closed subspace of B(H,H) where B(H,H) is the set of all bounded linear operators defined on H. [3]

3. (a) Define normal operator on a Hilbert space. If T_1 and T_2 are normal operators defined on a Hilbert space H such that either commutes with the adjoint of the other. Prove that $T_1 + T_2$ and T_1T_2 are normal operators on H. [4]

   (b) Let $T : H \to H$ be a self-adjoint operator where H is a Hilbert space. Show that $||T|| = Sup\{| < Tx, x > | : ||x|| = 1\}$. [4]

   (c) Prove that every self-adjoint operator is normal but converse may not be true. [2]

4. (a) Define compact linear operator on normed linear space and give an example for the same . [3]

   (b) If T is a normal operator defined on a Hilbert space H. Show that $Tx = \lambda x$ iff $T^*x = \bar{\lambda}x \; \forall x \in H$and for any scalar λ. [3]

   (c) Prove that for any nonempty subset M of a Hilbert space H, the span of M is dense in H iff $M^\perp = \{\theta\}$ where $M^\perp$ denotes the orthogonal complement of M. [4]

5. (a) Show that the spectrum $\sigma(T)$ of a bounded linear operator T defined on a complex Banach space is compact(assume $\sigma(T) \neq \phi$). [5]

   (b) Prove that residual spectrum $\sigma_r(T)$ of a bounded self-adjoint linear operator $T : H \to H$ where H is a complex Hilbert space is empty. [3]

   (c) Let T be a bounded linear operator defined on a Hilbert space H into itself. Then prove that $\sigma(T^*) = \{ \ k \in \mathbb{C} : \overline{k} \in \sigma(T) \ \}$. [2]

6. (a) Define Banach algebra and give an example for the same . [3]

   (b) Let A be a complex Banach algebra with identity e. If $x \in A$ satisfies $||x|| < 1$ then prove that, $(e - x)$ is invertible and $(e - x)^{-1} = e + \sum_{i=1}^{\infty} x^i$. [4]

   (c) Let $T : H \to H$ be a self-adjoint operator where H is a complex Hilbert space. Show that the eigen vectors corresponding to different eigen values of T are orthogonal. [3]