M. Sc. Examination-2025

Semester-II

Mathematics

Course: MMC-21

(Functional Analysis)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any <u>four</u> questions

1. (a) Define compact metric space and prove that $\mathbb{R}$ is not a compact metric space. [3]
   (b) Prove that every compact subset in a metric space is closed and bounded. Does the converse hold ? Support your answer. [4]
   (c) Let A be a compact subset in a metric space (X, d). Show that for every subset B of X there exists a point $x_0 \in A$ such that $d(x_0, B) = d(A, B)$. [3]

2. (a) State Riesz Lemma and hence show that if a closed unit ball in a metric space X is compact then X is finite dimensional. [4]
   (b) Show that $C[a, b]$ is a Banach space with respect to a norm to be defined by you. [4]
   (c) Show that $\mathbb{R}^2$ is a normed linear space with respect to $|| \ ||$ defined by $||x|| = max\{\frac{|x_1|}{a}, \frac{|x_2|}{b}\}$ where $a, b > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$. [2]

3. (a) Show that every finite dimensional normed linear space is complete. [3]
   (b) Prove that dual space X^* of any normed linear space X is a Banach space. [5]
   (c) If $T_1, T_2 : X \to Y$ are bounded linear operators where X and Y are normed linear spaces then prove that
   (i) $||T_1 + T_2|| \leq ||T_1|| + ||T_2||$ and (ii) $||T_1T_2|| \leq ||T_1||||T_2||$. [2]

4. (a) State and prove Closed Graph Theorem. [7]
   (b) Let $T : C[0, 1] \to \mathbb{R}$ defined by $T(x) = \int_0^1 xdt$, $x \in C[0, 1]$. Show that T is a bounded linear functional. [3]

5. (a) State and prove Bessel's Inequality in Hilbert space. [4]
   (b) Prove that $\mathbb{C}^n$ is a Hilbert space w.r.t. an inner product to be defined by you. [4]
   (c) Show that $l_p(p \geq 1,\ p \neq 2)$ is not a Hilbert space. [2]

6. (a) State and prove Riesz representation theorem for a bounded linear functional defined on a Hilbert space. [5]
   (b) Show that a normed linear space $(X, || \ ||)$ is a Banach space iff $\{x \in X : \ ||x|| = 1\}$ is complete. [3]
   (c) Prove that closure of a convex set in a normed linear space is convex. [2]

M. Sc. Examination-2025

Semester-II

Mathematics

Course : MMC-22 (Topology)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any **four** questions

1. (a) Prove Schröder–Bernstein theorem for cardinalities. [4]
   (b) For cardinal numbers u and v show that $2^{u+v} = 2^u.2^v$. [3]
   (c) State and prove Axiom of choice using Zorn's lemma. [3]

2. (a) State and prove the necessary and sufficient conditions for a collection $\mathcal{B}$ to be a basis for a topology. [4]
   (b) Give an example of a topology on a 3-element set that is neither discrete nor indiscrete, and justify. [3]
   (c) Define T_2 and T_1 spaces. Prove that every T_2 space is T_1, but converse is not true. [3]

3. (a) Give an example with justification of an open map that is not continuous. [3]
   (b) Prove that a map f is closed if and only if $\overline{f(A)} \subseteq f(\overline{A})$ for any A. [4]
   (c) Show that $A^o \cup B^o \subset (A \cup B)^o$ and give an example to show that the inclusion can be strict. [3]

4. (a) Show that every compact subspace of a T_2 space is closed and show, by example, that this may fail if the space is not T_2. [3+3]
   (b) Show that a space is regular if and only if $\forall x \in X$ and $\forall$ open set $V \subset X$ containing x, $\exists$ an open set V_x containing x such that $\overline{V_x} \subset V$. [4]

5. (a) When is a topological space called metrizable ? Give an example, with justification, of a topological space that is not metrizable. [1+3]
   (b) Show that every subspace of a T_3 space is T_3. [3]
   (c) Show that every metric space is normal. [3]

6. (a) Show that the union of any family of connected sets with nonempty intersection is connected. [3]
   (b) Define component of a topological space, prove that every component of a topological space is closed. [3]
   (c) Examine, with justification, whether the following spaces are connected:
      1. $\mathbf{R}$ with the usual topology,
      2. $\mathbf{R}$ with the lower limit topology,
      3. $\mathbf{R}$ with the co-countable topology,
      4. $\mathbf{R}$ with the discrete topology.

   [4]

M.Sc. Examination-2025

Semester-II

Mathematics

Paper: MMC-23

(Abstract Algebra)

Time: Three Hrs **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer any ***Four*** questions.

1. (a) Can the group D_4 be expressed as an internal direct product of two of its proper subgroups? Justify your answer. [3]
   (b) Find the number of elements of order 5 in $\mathbb{Z}_5 \times \mathbb{Z}_{25}$. [3]
   (c) For two relatively prime positive integers m, n show that $U(mn) \simeq U(m) \times U(n)$. [4]

2. (a) Let G be a group of order p^n and S be a G-set, where p is a prime number. Show that $|S| \equiv |\{a \in S : ga = a \text{ for all } g \in G\}| (\mod p)$. [3]
   (b) Find the class equation of Q_8. [3]
   (c) Find all 2-subgroups of A_4. [2]
   (d) Show that every group of order $2p$, where p is an odd prime is simple. [2]

3. (a) State and prove Sylow's second theorem on finite groups. [1+4]
   (b) Show that there does not exits any simple group of order p^2q for two distinct primes p and q. [4]
   (c) Classify all finite Abelian groups of order 60. [1]

4. (a) Show that every prime element in an integral domain is irreducible. Give an example with proper justifications that the converse may not hold. [2+3]
   (b) How many units are there in the ring $\mathbb{Z}_8[x]$? Justify your answer. [2]
   (c) Check whether the ideal $< 2, x >$ principal or not in the polynomial ring $\mathbb{Z}[x]$? Does there exist an Euclidean valuation in $\mathbb{Z}[x]$? Justify your answer. [1+1]

5. (a) Show that every principal ideal domain is a unique factorization domain. [5]
   (b) Find the GCD of 2 and $1 + i\sqrt{5}$ in $\mathbb{Z}[i, \sqrt{5}]$ [3]
   (c) Show that $\mathbb{Z}[\sqrt{2}]$ has no unitm in between 1 and $1 + \sqrt{2}$. [1+1]

6. (a) State Eisenstein's criterion of irreducibility of a polynomial over a unique factorization domain. Hence, prove or disprove that $\frac{Q[x,y]}{<y^2-x>}$ is a field. [1+3]
   (b) Find all values of n for which $x^3 + nx + 2$ is irreducible over $\mathbb{Z}$. [3]
   (c) Factorize (if possible) the polynomial $x^4 + 1$ in $\mathbb{Z}_5[x]$. [3]

M. Sc. Examination-2025

Semester-II

Mathematics

Course: MMC-24 (Classical Mechanics)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Obtain the equation of motion of a mass suspended from a significant height (considering the Earth's rotation about its axis). Assuming the oscillation is within a close neighborhood of its stable equilibrium, show that the mass oscillation rotates in the horizontal plane. Find the sense (clockwise or anticlockwise) of rotation if the point of suspension is on the Earth's northern hemisphere. [10]

2. A symmetric rigid body is rotating freely. Derive the evolution equation for the angular velocity $\vec{\omega}(t)$. Verify whether or not $|\vec{\omega}(t)|$ changes with time. Examine whether $\vec{\omega}(t), \vec{\Omega}(t)$ and the axis of symmetry of the symmetric rigid body lie on a plane throughout the motion. [10]

3. Derive the equation of motion of a particle relative to a frame of reference fixed at a point of colatitude λ on the Earth's surface (state the assumptions invoked during your derivation). A particle is thrown with a high velocity v_0 in the nearly horizontal direction towards the east from a point on the Earth's surface. If the air resistance is neglected, prove that the particle will always be deflected towards the equator. [10]

4. Derive the expression for the kinetic energy of a holonomic system having n degrees of freedom. Show that the kinetic energy of a scleronomic holonomic system is a homogeneous function of generalized velocities of degree 2. Hence, the Lagrange equation of a scleronomic holonomic system is exhibited as a system of coupled ODEs. Determine their order. [10]

5. **a)** State and prove the theorem of Legendre dual transformation. Hence or otherwise, derive Hamilton's equation of motion.

   b) Explain the process of canonical transformations in classical mechanics. [5+5]

6. **a)** Describe normal coordinates and normal frequency of oscillations of a mechanical system around the close vicinity of one of its stable equilibrium state.

   b) State the basic principles of quantum mechanics and canonical quantization rules. Derive the Schrödinger equation for the description of microscopic motion of a particle moving under the influence of force described by the potential energy function $V(\vec{r})$. [5+5]

Use separate answer script for each unit

M. Sc. Examination-2025

Semester-II
Mathematics
Paper : MMC-25 (New Syllabus)
(Solid Mechanics and Dynamical System)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I [Solid Mechanics (Marks: 20)]

Answer *any two* questions.

1. (a) Define strain vector. Obtain the relation between strain vector and strain tensor. [2+3]

   (b) The displacement in an elastic solid is given as follows

$$\begin{aligned} u_1 &= \epsilon(X_1 + 2X_2 + 3X_3) \\ u_2 &= \epsilon(-2X_1 + X_2) \\ u_3 &= \epsilon(X_1 + 4X_2 + 2X_3) \end{aligned}$$

   where ϵ is a small quantity. Calculate dilatation, rotation, shear and principal strains. [5]

2. (a) Prove that the stress vector at a point on any arbitrary plane surface is a linear function of three stress vectors acting on any three mutually perpendicular planes through that point. [5]

   (b) The state of stress at a point is given by
   $\tau_{ij} =$

$$\begin{bmatrix} a & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

   where a is a constant. Determine a such that there is at least one plane through the point in such a way that resultant stress on that plane is zero. Determine the d.cs. of the normal to that plane. [5]

3. (a) State the principles of conservation of mass in Lagrangian and Eulerian methods. Write down their corresponding equations of continuity and prove their equivalence. [2+1+2]

   (b) Write down the stress equation of equilibrium. Given the following stress distribution $\tau_{ij} =$

$$\begin{bmatrix} x_2 & -x_3 & 0 \\ -x_3 & 0 & -x_2 \\ 0 & -x_2 & T \end{bmatrix}$$

   find T such that stress distribution is in equilibrium with body force $\overrightarrow{F} = -g\overrightarrow{e_3}$. [1+4]

Unit-II [Dynamical System (Marks: 20)]

Answer *any two* questions.

1. (a) Find all fixed points and their stabilities of $f(x) = x^2 - 1$, $x \in \mathbb{R}$. [2]

   (b) For $f(x) = -\frac{3}{2}x^2 + \frac{5}{2} + 1$, $x \in \mathbb{R}$, show that $\{0, 1, 2\}$ is an unstable 3-cycle . [3]

   (c) State Poincaré-Bendixson theorem. Show that the system given by $\dot{x} = x - y - x(x^2 + 2y^2)$, $\dot{y} = x + y - y(x^2 + 2y^2)$ admits at least one periodic orbit in a region R. Specify the region R. [1+4]

2. (a) Define a logistic map. Prove that the logistic map $f(x) = rx(1 - x)$ has a 2-cycle for $r > 3$. [1+4]

   (b) Define Schwarzian Derivative and find the Schwarzian derivative of a function $f(x) = \sin x$ at $x = 0$. [2]

   (c) Suppose $f(x)$ is a polynomial such that all the roots of $f'(x)$ are real and distinct. Then show that $Sf(x) < 0$. [3]

3. (a) Define the Tent map $T(x)$. Also find the fixed points of $T^2(x)$. Show that the set of points $\{\frac{2}{5}, \frac{4}{5}\}$ forms a periodic 2-cycle of the tent map $T(x)$. [1+2+2]

   (b) State and Prove Dulac's Criteria. [1+2]

   (c) Find the potential function of the dynamical system

$$\dot{x} = 2xy + y^3, \qquad \dot{y} = x^2 + 3xy^2 - 2y.$$

[2]

M.Sc. Examination, 2025

Semester-II

Mathematics

Paper: MMC-26 (New & Old Syllabus)

(Numerical Analysis)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Suppose $f \in C^{2n+2}[a,b]$ and $x_0, x_1, x_2, \cdots x_n$ are distinct points in $[a,b]$. Let $H_{2n+1}(x)$ be the Hermite interpolating polynomial for f with respect to these points. Show that the error in Hermite interpolation is given by

$$R(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\prod_{i=0}^{n}(x-x_i)^2, \ \forall x \in [a,b], \ a < \xi < b.$$

[5]

(b) Find the cubic spline curve that passes through $(0,0)$, $(1,0.5)$, $(2,2)$ and $(3,1.5)$ with the vanishing curvature end conditions. [5]

2. (a) If $P_n(x)$ is any monic polynomial of degree n and $\tilde{T}_n(x) = \frac{1}{2^{n-1}}T_n(x)$ is the monic Chebyshev polynomial, then show that $\max_{x\in[-1,1]} |\tilde{T}_n(x)| < \max_{x\in[-1,1]} |P_n(x)|$. [5]

(b) A person runs the same race track for five consecutive days and is timed as follows:

day (x)	1	2	3	4	5
time (y)	15.30	15.10	15.00	14.50	14.00

Find the least square fit to the above data using the function $a + \frac{b}{x} + \frac{c}{x^2}$.

[5]

3. (a) What do you mean by Romberg's quadrature? Derive Romberg's quadrature formula for evaluating $I = \int_a^b f(x)dx$ by Simpson's one-third rule. Also mention the stopping criterion. [1+4+1]

(b) Obtain an approximate value of the integral $I = \int_{-1}^{1} (1-x^2)^{\frac{1}{2}} \cos x \, dx$ by using the Gauss-Chebyshev three-point formula. [4]

4. (a) If $C_i^{(n)}$ be the Cotes' coefficients, then show that $C_i^{(n)} = C_{n-i}^{(n)}$, $i = 0, 1, 2, \cdots n$ [3]

(b) Outline the sequence of steps in the Shooting method for numerically solving two-point boundary value problems for ordinary differential equations. [2]

(c) Show that the finite difference method for approximating the boundary value problem $y'' + p(x)y' + q(x)y = r(x), a < x < b,$ $p(x)$, $q(x)$, $r(x) \in C[a,b]$ with boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ leads to a tri-diagonal system. [5]

5. (a) If $f(x,y)$ has continuous partial derivatives upto sufficient number of times for $(x,y) \in D$ where, $D = \{(x,y) : a \le x \le b, -\infty < y < \infty\}$ then show that the grid error E for a single-step method in solving the initial value problem $\frac{dy}{dx} = f(x,y)$, $y(a) = y_0$ is given by

$$E = \frac{Kh^p}{M}\left(e^{(b-a)M} - 1\right),$$

where, K, M are positive constants, p is the order of the single-step method. [5]

(b) Solve the heat conduction equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$, $(\alpha > 0)$, by FTCS scheme with initial condition $u(x,0) = 0.8 \sin \pi x$, $0 \le x \le 1$, and boundary conditions $u_x(0,t) = u_x(1,t) = 0$, $(t > 0)$. Taking $\alpha = 1$, $r = \frac{1}{2}$, $\Delta x = \frac{1}{4}$, compute the solution for the first two time step. [5]

6. (a) Find the solution at $x = 0.2$ for the initial value problem $y' = x^2 - 3y, y(0) = 2$ using Adams-Moulton second order method with $h = 0.1$. [4]

(b) What do you mean by stability of a finite difference scheme? Investigate the stability of the Crank-Nicholson scheme for approximating the heat-conduction problem $u_t = \alpha u_{xx}$, $u(x,0) = f(x), 0 < x < L$ and $u(0,t) = u(1,t) = 0, \forall\, t > 0$ at the $(i,n)^{\text{th}}$ point with α being positive constant. [1+5]