Use separate answer script for each unit

B. Sc. (Honours) Examination-2022

Semester-IV
Mathematics (Major)
Paper : CC-9 (Mechanics-I)
(Dynamics of a Particle and Statics)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I [Dynamics of a Particle (Marks: 30)]

Answer *any three* questions.

1. Define simple harmonic motion. A particle moves from rest in a straight line under an attractive force $\mu\times$ (distance)$^{-2}$ per unit mass to a fixed point on the line. Show that if the initial distance from the center of force be $'2a'$, then the distance will be $'a'$ after a time $\left(\frac{\pi}{2}+1\right)\sqrt{\frac{a^3}{\mu}}$. [2+

2. Obtain the expressions for tangential and normal accelerations of a particle moving in a plane curve. [1

3. (a) A particle describes an equiangular spiral $r = ae^{m\theta}$ with constant velocity. Find the components of velocity and acceleration along the radius vector and perpendicular to it . [

   (b) A particle describes a path with an acceleraton $\frac{\mu}{y^3}$ which is always parallel to the axis of y and directed toward the x-axis. If the particle be projected from a point $(0, a)$ with the velocity $\frac{\sqrt{\mu}}{a}$ parallel to x-axis, then show that the path described by is a circle. [5

4. (a) If the central orbit be an ellipse under a force towards the centre, then find the law of force. Find also the velocity at any point of the orbit. [5

   (b) A particle moves with a central acceleration $\mu\left(r+\frac{a^4}{r^3}\right)$ being projected from an apse at a distance a with a velocity $2\sqrt{\mu}a$. Prove that it describes the curve $r^2\left(2+\cos\sqrt{3}\theta\right) = 3a^2$. [5

5. Find the equation of a trajectory when a particle is projected with a velocity u at an angle α to the horizon in a medium whose resistance is $mk\times$ (velocity). [10

Unit-II [Statics (Marks: 30)]

Answer *any three* questions.

1. Define astatic equilibrium. Find the condition for astatic equilibrium of a system of coplanar forces acting at different points of a rigid body. [2+8

2. Three forces P, Q, R act along the sides of a triangle formed by the lines $x+y=3$, $2x+y=1$ and $x-y=-1$. Find the equation of the line of action of the resultant. [10

3. Define 'forces of constraints'. Use the principle of virtual work to find the conditions of equilibrium of a system of coplanar forces acting on a rigid body. [2+8

4. (a) Obtain the expression for virtual work done by the tension in a string. [5

   (b) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that $\tan\phi = \frac{3}{8} + \tan\theta$. [5

5. (a) Find the conditions of equilibrium of a system of non-coplanar forces acting at different points of a rigid body. [5

   (b) Six forces, each equal to P, act along the edge of a cube, taken in order, which do not meet a given diagonal. Show that their resultant is a couple of moment $2\sqrt{3}.Pa$, where a is the edge of the cube. [5

B.Sc.(Honours) Examination-2022

Semester-IV

Mathematics

Course: SECMA-2

(Tensor Calculus)

Time: Two Hours **Full Marks: 25**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meaning.

Answer ***any five*** questions.

1. Define scalar, vector and tensor fields. The relations among components $A(i, j, k, l)(\tilde{u})$ in two coordinate systems $\tilde{u}(=(u^1, u^2, \cdots, u^n))$ and $\tilde{\bar{u}}(=(\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^n))$ are
$$\bar{A}(\alpha, \beta, \gamma, \delta)(\tilde{\bar{u}}) = A(\alpha, \beta, \gamma, \delta)(\tilde{u}).$$
Determine whether $A(i, j, k, l)(\tilde{u})$ are tensor. Give their order and rank. Write down the explicit expression for $A_{pq}A^{qr}$ following the rule of summation convention. [1+1+1+1+1]

2. Assuming a point in space can be described by two sets of variables $\tilde{u}(=(u^1, u^2, \cdots, u^n))$ and $\tilde{\bar{u}}(=(\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^n))$ uniquely, prove that du^p and $\frac{\partial\phi}{\partial u^k}(\tilde{u})$ are tensor fields. Determine their natures (covariant/contravariant). [2+2+1]

3. If A_r^{pq} and B_s^t are tensor fields, verify the outer product $A_r^{pq}B_s^t$ is a tensor field or not. Show that contraction of the outer product of tensor fields A^p and C_r is a scalar field. Find the rank of the inner product of the tensor fields C_r^p and B_t^{qs}. [2+2+1]

4. A quantity $B(i, j, k)$ is such that $B(i, j, k)C_i^{jl} = A_k^l$, where C_i^{jl} is an arbitrary tensor field. Prove that $B(i, j, k)$ is a tensor field. Determine its rank. [4+1]

5. If $ds^2 = g_{pq}du^p du^q$ is an invariant, prove that g_{pq} is a tensor. Determine its rank and verify whether it is symmetric. Find g_{pq} for the cylindrical polar coordinates (u^1, u^2, u^3). [3+1+1]

6. Define Christoffel's symbol of first and second kinds. Prove that the Christoffel symbol of the first kind is symmetric in the first two indices, and the second kind is symmetric for indices appearing in the subscript. Prove the formula $\Gamma_{pq}^p = \frac{\partial}{\partial u^q}\ln\sqrt{g}$. [1+2+2]

7. Prove that $\frac{\partial}{\partial u^q}\Phi(\tilde{u})$ is a tensor for any scalar field $\Phi(\tilde{u})$. Check whether $\frac{\partial}{\partial u^q}A^p(\tilde{u})$ is a tensor field. Define covariant derivative. Find the covariant derivative of $g_{st}(\tilde{u})$ with respect to the variable u^i. [1+1+1+2]

8. For any tensor field $A_q(\tilde{u})$, establish the formula $A_{p,qr} - A_{p,rq} = R_{pqr}^s A_s$. Here $B_{p,q}$ stands for the covariant derivative of the tensor field $B_p(\tilde{u})$ with respect to u^q. [5]

B.Sc. (Honours) Examination, 2022

Mathematics

Semester - IV

Paper/Course CCMA - 10

Probability Theory

Time: Three Hours **Full Marks: 60**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meaning.

Answer *any six* questions

1. Explain the concept of statistical regularity. State clearly the frequency definition of probability of an event A connected with a random experiment E. Explain the real strength of this definition. Hence define the conditional probability of the event B on the hypothesis that event A has already occurred. [4+2+2+2]

2. (a) If $\{A_n\}$ is a monotonic sequence of events, then show that

$$P(\lim A_n) = \lim P(A_n).$$

[4]

(b) A card is drawn at random from each of two well-shuffled packs of cards. What is the probability that at least one of them is a queen of spades? [3]

(c) What is the probability that a bridge hand contains four aces? [3]

3. (a) State Bayes' theorem. There are three urns containing respectively 1 white and 2 black balls, 2 white and 1 black balls, 2 white and 2 black balls. One ball is transferred from the first to the second urn; then one ball is transferred from the second to the third urn; finally, one ball is drawn from the third urn. Find the probability that the ball drawn is white. [2+4]

(b) Consider three events A, B, C. If A and B are independent and B and C are independent, then does it follow that events A and C are independent? [4]

4. (a) Define Poisson sequence of trials. Three marksmen can hit a target with probabilities $1/2, 2/3, 3/4$ respectively. They shoot simultaneously and two hits were registered. Compute the probability that each of the three marksmen misses the target. [2+3]

(b) Define Markov chain. In a Markov chain of coin tossings, the first toss is given to be fair and the transition probabilities of $H \to H, H \to T, T \to H, T \to T$ are respectively $1/2, 1/2, 0, 1$. Find the probability of head(H) in the third toss and tail(T) in the fourth toss. [1+4]

5. (a) Define a random variable X. Define distribution function F of a random variable X. For any fixed point a, compute $F(a-0)$ and $F(a+0)$. [1+2+1+1]

(b) If X is a normal (m, σ) variate, then show that

$$P(| X - m | > a\sigma) = 2[1 - \Phi(a)]$$

where $\Phi(x)$ denotes the standard normal distribution function and a is a fixed point. [5]

6. (a) Show that the function $| x |$ in $(-1, 1)$ is a possible probability density function and hence find the distribution function. [4]

(b) If X is uniformly distributed over the interval $(-2, 2)$ then find the distribution of the random varible $\min\{X, 1\}$. [3]

(c) If X is normal $(0, 1)$ then find the distribution of e^X. [3]

7. (a) Define $k-$th central moment μ_k of a random variable X. Compute the same for normal (m, σ) variate. [1+2]

(b) The probability density of a continuous distribution is given by

$$f(x) = \frac{3}{4}x(2-x);\ 0 < x < 2.$$

Compute the mean and variance. [2+2]

(c) Define mode of a distribution. Find the mode of a Binomial $(4, 0.25)$ variate. [1+2]

8. (a) Show that if X is a $\gamma(\frac{1}{2}n)$ variate then $2X$ has a chi-square distribution with n degrees of freedom. Compute the mean, variance and mode of $\chi^2(n)$ variate. [2+3]

(b) For the $t-$distribution with n degrees of freedom, prove that the variance exists only for $n > 2$ and its value is $n/(n-2)$. [5]

9. (a) Define joint distribution of a pair of random variables. Define their marginal distributions. Prove that if X and Y are independent then

$$P(X = b, Y = d) = P(X = b)P(Y = d).$$

[2+2+3]

(b) X and Y are independent variates, each uniformly distributed over the interval $(0, 1)$. Find the probability that the greater of X, Y is less than a fixed number $k(0 < k < 1)$. [3]

10. (a) Define convergence in probability. If X_n is a Binomial(n, p) variate, then show that

$$\frac{X_n}{n} \to p \text{ in probability as } n \to \infty.$$

[1+4]

(b) If $X_1, X_2, \cdots X_n, \cdots$ be a sequence of independent and identically distributed random variables each with mean m and variance $\sigma^2 < \infty$ for $i = 1, 2, \cdots$ then show that

$$\lim_{n\to\infty} P(|\bar{S}_n - m| \geq \epsilon) = 0$$

for every $\epsilon > 0$. [5]

Use separate answer script for each unit

B. Sc. (Honours) Examination-2022

Semester-IV (CBCS)
Mathematics
Paper: CCMA 8
(Analysis-IV and Differential Equations-II)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Analysis-IV)
(Full Marks: 30)
Answer *any three* questions.

1. (a) Prove that limit of a function of two variables if exists, is unique. [3]
   (b) Using $\epsilon-\delta$ definition, show that $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}=0$. [3]
   (c) Prove that if $f(x,y)$ is continuous at a point (a,b) and $f(a,b)\neq 0$ then $f(x,y)$ has the sign of $f(a,b)$ in some neighbourhood of the point (a,b). [4]
2. (a) State and prove a sufficient condition for continuity of a function $f(x,y)$ at a point (a,b). [4]
   (b) For the function $f(x,y)=\begin{cases} xy & \text{for } |x|\geq|y| \\ -xy & \text{for } |x|<|y| \end{cases}$, show that $\frac{\partial^2 f}{\partial x\partial y}(0,0)\neq\frac{\partial^2 f}{\partial y\partial x}(0,0)$. [3]
   (c) Using Euler's theorem, if $u=tan^{-1}\frac{x^3+y^3}{x^2+y^2}$ then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}sin2u$. [3]
3. (a) State and prove a sufficient condition for differentiability of a function $f(x,y)$ at a point (a,b). [5]
   (b) State Young's theorem for equality of two mixed second order partial derivatives of a function $f(x,y)$ at a point (a,b).
   Show that for the function $f(x,y)=\begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{for } (x,y)\neq(0,0) \\ 0 & \text{for } (x,y)=(0,0) \end{cases}$, $\frac{\partial^2 f}{\partial x\partial y}(0,0)=\frac{\partial^2 f}{\partial y\partial x}(0,0)$ but the conditions of Young's theorem are not satisfied. [5]
4. (a) State Implicit function theorem. Show that $F(x,y)=x^2+2xe^y+y=0$ determines a unique function $y=f(x)$ about $(0,0)$. Also find $\frac{dy}{dx}$ at $(0,0)$. [4]
   (b) State and prove mean value theorem for a function of two variables. [3]
   (c) Using Taylor's theorem, expand $f(x,y)=xy^2+3x-2$ in powers of $(x+1)$ and $(y-2)$. [3]
5. (a) Show that a necessary condition for $f(x,y)$ to have an extreme value at (a,b) is that $f_x(a,b)=0$ and $f_y(a,b)=0$, provided these partial derivatives exist. [3]
   (b) Show that $f(x,y)=y^2+x^2y+x^4$ has a minimum at $(0,0)$. [4]
   (c) Evaluate $\int\int_R(x^2+y^2)dxdy$ over R bounded by $y=x^2, x=2, y=1$. [3]

Unit-II (Differential Equations-II)
(Full Marks: 30)
Answer question number 1 and *any two* from the rest.

1. Answer *any five* questions.
   (a) If a transformation $y=uv$ transforms the given ordinary differential equation (ODE) $\phi(x)\frac{d^2y}{dx^2}-4044\frac{d\phi}{dx}\frac{dy}{dx}+\psi(x)y=0$ into the normal form $\frac{d^2v}{dx^2}+\xi(x)v=0$, then evaluate u. Share your observation how v can be ontained.
   (b) Given the solution $y_1(x)=\frac{1}{x-1}$, find the second linearly independent solution of the ODE $(x^2-x)\frac{d^2y}{dx^2}+(3x-1)\frac{dy}{dx}+y(x)=0,\ x\neq 0,1$.
   (c) Show that the equation $(x^2+2y^2\frac{dy}{dx})\frac{d^2y}{dx^2}+2y(\frac{dy}{dx})^3+2(x+ycosx)\frac{dy}{dx}-(sinx)y^2=0$ is exact and find its first integral.
   (d) What do the solutions of the differential equation

$$\frac{dx}{3}=\frac{dy}{4}=\frac{dz}{5}$$

represent geometrically? What are the significance of the numbers in the denominators?

(e) Consider the following ordinary simultaneous differential equations:

$$\begin{aligned}(D+1)x+(D-1)y &= e^t,\\ (D^2+D+1)x+(D^2-D+1)y &= t^2,\ \ D^n \equiv \frac{d^n}{dt^n}.\end{aligned}$$

The general solution of the system contains no arbitrary constant - Justify the statement with reason.

(f) Find an integrating factor of the total differential equation

$$(2xz-yz)dx+(2yz-zx)dy-(x^2-xy+y^2)dz=0.$$

Also, carry out your ideas if $z(x^2+y^2-xy)=0$ holds.

(g) Convert the following ODE into two first-order equations:

$$a_0(t)\frac{d^2y}{dt^2}+a_1(t)\frac{dy}{dt}+a_2(t)y=a_3(t);\ \ a_0(t)\neq 0$$

with $y=c_1$ and $\frac{dy}{dt}=c_2$ at $t=0$.

(h) How many solutions exist for the boundary value problem

$$\frac{d^2y}{dx^2}+y=0,\ \ 0<x<\pi$$

subject to $y(0)+y(\pi)=0=y'(0)+y'(\pi)$? Justify your answer.

[5 × 2=10]

2. (a) Transform the linear ODE

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

by the substitution $y(x)=z(x)\ exp[-\frac{1}{2}\int_0^x P(t)dt]$ into its normal form $\frac{d^2z}{dx^2}+I(x)z=0$ where $I(x)=Q(x)-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}$.

(b) Solve:

$$\frac{dx}{x(2y^4-z^4)}=\frac{dy}{y(z^4-2x^4)}=\frac{dz}{z(x^4-y^4)}.$$

[5+5]

3. (a) Factorise the operator on the left hand side of $[xD^2+(x-2)D-2]y=x^3e^{2022x}$, $D^n\equiv\frac{d^n}{dx^n}$ and hence solve it.

(b) Solve the simultaneous differential equations:

$$\begin{aligned}4\frac{dx}{dt}+9\frac{dy}{dt}+11x+3y &= e^t,\\ 3\frac{dx}{dt}+7\frac{dy}{dt}+8x+24y &= e^{2t}.\end{aligned}$$

[5+5]

4. (a) Verify that the following total differential equation is integrable and hence solve the equation:

$$3x^2(y+z)dx+(z^2-x^3)dy+(y^2-x^3)dz=0.$$

(b) Solve the following system of two linear homogeneous differential equations with constant coefficients using eigenvalues and eigenvectors of the matrix:

$$\begin{aligned}\frac{dx}{dt} &= 7x+y,\\ \frac{dy}{dt} &= 7y,\\ x(t=0) &= 7,\ \ y(t=0)=1.\end{aligned}$$

[5+5]