**Five Year Integrated M.Sc. Examination, 2024**
**Semester-I**
**Subject: Mathematics I**
**Course Code: MT-1-1-1**

**Time: 3 Hours** **Full Marks: 60**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Attempt **Question Number 1** and any three from the rest.

1. Choose the correct answer from the following:
(i) Suppose $X$ and $Y$ are two finite sets such that $|X| = 5$ and $|Y| = 4$. The number of functions that can be defined from $X$ into $Y$ is
(a) $5^4$ (b) $4^5$ (c) 20 (d) none
(ii) The number of bijective functions on a set X having 5 elements is
(a) 5 (b) $5^5$ (c) $2^5$ (d) none
(iii) Using $\epsilon - \delta$ definition, the value of $\delta$ in $\lim_{x\to 3}(3x + 2) = 11$ is
(a) $\epsilon/2$ (b) $\epsilon/3$ (c) $\epsilon$ (d) none
(iv) Where is the function $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?
(a) $\mathbb{R}$ (b) $\mathbb{R}^+ - \{\pm 1\}$ (c) $\pm 1$ (d) none
(v) Given $g(x) = f\left(\dfrac{2x-1}{x+1}\right)$ and $f'(0.5) = 1$, then $g'(1)$ is
(a) 0.5 (b) 2.0 (c) 2.25 (d) none
(vi) If $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2 - 25}$, the sketch of the domain consists of
(a) the entire $xy$ plane (b) the points not on the circle $x^2 + y^2 = 25$ (c) the points on and inside the circle $x^2+y^2 = 25$ (d) the points on and outside the circle $x^2+y^2 = 25$
(vii) If $\dfrac{d}{dx}f(x) = g(x)$ and if $h(x) = x^2$ then $\dfrac{d}{dx}f(h(x))$ is
(a) $g(x^2)$ (b) $2xg(x)$ (c) $g'(x)$ (d) $2xg(x^2)$
(viii) If a function $f(x, y)$ is discontinuous at a point (a,b) then
(a) $\lim_{(x,y)\to(a,b)} f(x, y)$ exist and equal to $f(a, b)$ (b) $f(x, y)$ is not differentiable at $(a, b)$ (c) $f(x, y)$ is differentiable at $(a, b)$ (d) None of these.
(ix) The set $f = \{(a, b), (b, c), (c, d)\}$ where $A = B = \{a, b, c, d\}$. Is f a function
(a) yes (b) no $2\times 9=18$

2. Prove or disprove. Justify your answer briefly.
(a) Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x^2 - 5$. The value of $f(f^{-1}(10)$ is 10.
(b) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x|x|, x \in \mathbb{R}$ is neither injective nor surjective.

(c) The relation $\rho = \{(1,1),(2,2),(3,3),(4,4),(3,2),(4,3),(3,4),(2,3)\}$ on $A = \{1,2,3,4\}$ is an equivalence relation.
(d) The intersection of two equivalence relations is again an equivalence relation.
(e) If a function $f$ is differentiable at $a$ then it is continuous there.
(f) Every bijective function is an onto function but the converse is not true.
(g) Every relation is a function. $2\times 7=14$

3. (i) Define level curve and graph of a function of two variables. Sketch the level curves of the function $g(x,y) = \sqrt{4x^2+y^2}$.
(ii) Find the limit, if it exists, or show that the limit does not exist:
(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y^2}{x^5+y^7}$ (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy\cos(y/x)}{3x^2+y^2}$
(iii) Find the equation of the tangent plane to the surface $f(x,y) = 4x^2 - y^2 + 2y$ at (-1, 2, 4). Hence find the linear approximation. (2+2)+(3+3)+(1+3)=14

4. (i) Determine whether the function $f(x,y) = \dfrac{\sqrt{2021x^2y^2 - 2022\pi y^4 + 2023x^3y}}{2025x^2+2024y^2}$ is homogeneous or not. If it is so, find the degree. Verify Euler's first theorem for the function $f(x,y)$.
(ii) Show that the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq 0, \\ 0, & (x,y)=0. \end{cases}$$

is continuous at (0, 0). From definition, show that $f_x(0,\ 0)$ and $f_y(0,\ 0)$ both exist.
(iii) What do you mean by harmonic function. (2+1+3)+(3+2+2)+1=14

5. (i) Let a function $f$ be defined as

$$f(x,y) = \begin{cases} \dfrac{xy\left(1-\frac{x^2}{y^2}\right)}{\left(1+\frac{x^2}{y^2}\right)}, & (x,y)\neq 0, \\ 0, & (x,y)=0. \end{cases}$$

Prove that $f$ is differentaible at (0, 0). Show that $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.
(ii) State Clairaut's theorem on derivatives. Verify Clairaut's theorem for $f(x,y) = x\sin(x+2y)$. (1+2)+(4+3.5+3.5)=14

6. (i) Find $\lim_{x\to 0}\left(\dfrac{1}{\tan x} - \dfrac{1}{x}\right)$.
(ii) Consider, $f'$ is continuous. Under what condition, L'Hospital rule will be applicable for the limit $\lim_{x\to 0}\dfrac{f(2+3x)+f(2-5x)}{x}$. Hence, evaluate the limit where

$f'(2)$=7.
OR
For what values of $a$ and $b$, is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3}+\frac{a}{x}+\frac{b}{x^2}\right)=0.$$

(iii) Use the squeeze theorem to show that

$$\lim_{x\to 0}\sqrt{x^3+x^2}\sin\frac{\pi}{x}=0.$$

(iv) Does the limit $\lim_{z\to 0}\dfrac{z}{\bar{z}}$ exist? Justify. 3+(2+2)+3+4=14